Chapter 9

Unitary Groups and SU(N)*

The irreducible representations of SO(3) are appropriate for describing the degeneracies of states of quantum mechanical systems which have rotational symmetry in three dimensions. But there are many systems for which operations on classical coordinates must be supplemented by operations on "internal" degrees of freedom which have no classical analogue. For example, the Stern–Gerlach experiment showed that electrons are endowed with an internal degree of freedom called "spin" which has the properties of an angular momentum. The two spin states are therefore inconsistent with the dimensionalities of the irreducible representations of SO(3), so another group—SU(2)—must be used to describe these states. Since, as we will show in Section 9.2, SU(2) is locally isomorphic to SO(3), we can define a total spin S in an abstract three-dimensional space, analogous to the total angular momentum in real space. In particle physics, unitary symmetry was used to describe the approximate symmetry (called isospin) of neutrons and protons and, more recently, to describe particle spectra within the framework of the quark model.

In this chapter, we introduce unitary groups and their irreducible representations in a similar manner to which we developed SO(3). We begin by defining unitarity in terms of the invariance of an appropriate quantity and proceed to discuss the construction of irreducible representations of these groups in N dimensions. Higher-dimensional irreducible representations will be obtained with the aid of Young tableaux, which is a diagrammatic technique for determining the dimensionalities and the basis functions of irreducible representations derived from direct products.

9.1 SU(2)

As with orthogonal matrices, the unitary groups can be defined in terms of quantities which are left invariant. Consider a general complex transformation in two dimensions, $\boldsymbol{x}' = A\boldsymbol{x}$ which, in matrix form, reads:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where a, b, c, and d are complex, so there are eight free parameters. The determinant of this matrix is nonzero to permit the construction of inverses.

9.1.1 Unitary Transformations

Suppose we require the quantity $|x|^2 + |y|^2$ to be an invariant of such a transformation. Then,

$$\begin{aligned} |x'|^2 + |y'|^2 &= |ax + by|^2 + |cx + dy|^2 \\ &= (ax + by)(a^*x^* + b^*y^*) + (cx + dy)(c^*x^* + d^*y^*) \\ &= (|a|^2 + |c|^2)|x|^2 + (ab^* + cd^*)xy^* + (a^*b + c^*d)x^*y \\ &\quad + (|b|^2 + |d|^2)|y|^2 \\ &= |x|^2 + |y|^2 \end{aligned}$$

Since x and y are independent variables, this invariance necessitates setting the following conditions on the matrix elements:

$$|a|^2 + |c|^2 = 1, \qquad |b|^2 + |d|^2 = 1, \qquad ab^* + cd^* = 0$$

These four conditions (the last equation provides two conditions because it involves complex quantities) means that the original eight free parameters are reduced to four. These conditions are the same as those obtained by requiring the $A^\dagger A = 1$, so the determinant of the resulting matrix has modulus unity. These transformations are analogous to orthogonal transformations of real coordinates and, indeed, orthogonal transformations are also unitary. The group comprised of unitary matrices is denoted by U(2) and by U(N) for the N-dimensional case.

9.1.2 Special Unitary Transformations

If, in addition to the conditions above, we require that the determinant of the transformation is unity, the transformation matrix must have the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad |a|^2 + |b|^2 = 1 \tag{9.1}$$

There are now three free parameters and the group of these matrices is denoted by SU(2) where, as in our discussion of orthogonal groups, the 'S' signifies 'special' because of the requirement of a unit determinant.

9.2 Relation between SU(2) and SO(3)

9.2.1 Pauli Matrices

If the matrix elements of the general unitary matrix in (9.1) are expressed in terms of their real and imaginary parts, we can decompose this matrix into the components of a "basis." Thus, with $a = a_r + ia_i$ and $b = b_r + ib_i$, we have

$$\begin{aligned} U &= \begin{pmatrix} a_r + ia_i & b_r + ib_i \\ -b_r + ib_i & a_r - ia_i \end{pmatrix} \\ &= a_r \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + ia_i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \underbrace{b_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{ib_r \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}} + ib_i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

Thus, any 2×2 unitary matrix can be represented as a linear combination of the unit matrix and the matrices

$$\sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

These three (Hermitian) matrices are known as the **Pauli matrices**. They satisfy the following multiplication rules:

$$\begin{aligned} &\sigma_i^2 = I && (i = x, y, z) \\ &\sigma_i \sigma_j = -\sigma_j \sigma_i = i\varepsilon_{ijk}\sigma_k && (\{i, j, k\} = x, y, z) \end{aligned} \tag{9.2}$$

where I is the 2×2 unit matrix. These multiplication rules can be used to obtain a concise expression for the product of two matrices written as $\boldsymbol{a}\cdot\boldsymbol{\sigma}$ and $\boldsymbol{b}\cdot\boldsymbol{\sigma}$, where $\boldsymbol{a} = (a_x, a_y, a_z)$, $\boldsymbol{b} = (b_x, b_y, b_z)$, and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$:

$$(\boldsymbol{a}\cdot\boldsymbol{\sigma})(\boldsymbol{b}\cdot\boldsymbol{\sigma}) = (\boldsymbol{a}\cdot\boldsymbol{b})I + i(\boldsymbol{a} \times \boldsymbol{b})\cdot\boldsymbol{\sigma} \tag{9.3}$$

9.2.2 Infinitesimal Generators

Moreover, if we define matrices $X_i = -\frac{1}{2}i\sigma_i$, for $i = 1, 2, 3$, then the second of the multiplication rules in (9.2) yield the following commutation relations:

$$[X_i, X_j] = \varepsilon_{ijk} X_k$$

These are identical to commutators of the infinitesimal generators of SO(3) in (7.13). Thus, locally at least, there is an isomorphism between SO(3) and SU(2). Motivated by the discussion in Section 7.3, consider the matrix

$$U = \exp\left(-\tfrac{1}{2}i\varphi\boldsymbol{n}\cdot\boldsymbol{\sigma}\right)$$

where $\varphi\boldsymbol{n}$ is the axis-angle representation of a rotation (Section (8.3). Since the exponential of a matrix is defined by its Taylor series expansion, we have

$$U = \sum_{k=0}^{\infty} \frac{(-i)^n}{n!} (\tfrac{1}{2}\varphi)^n (\boldsymbol{n}\cdot\boldsymbol{\sigma})^n$$

From Equation (9.3), $(\boldsymbol{n}\cdot\boldsymbol{\sigma})^2 = I$, so

$$
\begin{aligned}
U &= I\sum_{k=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\tfrac{1}{2}\varphi\right)^{2n} - i(\boldsymbol{n}\cdot\boldsymbol{\sigma})\sum_{k=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\tfrac{1}{2}\varphi\right)^{2n+1} \\
&= \cos\left(\tfrac{1}{2}\varphi\right)I - i(\boldsymbol{n}\cdot\boldsymbol{\sigma})\sin\left(\tfrac{1}{2}\varphi\right) \\
&= \begin{bmatrix} \cos\left(\tfrac{1}{2}\varphi\right) - in_z\sin\left(\tfrac{1}{2}\varphi\right) & -(n_y + in_x)\sin\left(\tfrac{1}{2}\varphi\right) \\ (n_y - in_x)\sin\left(\tfrac{1}{2}\varphi\right) & \cos\left(\tfrac{1}{2}\varphi\right) + in_z\sin\left(\tfrac{1}{2}\varphi\right) \end{bmatrix}
\end{aligned}
\tag{9.4}
$$

This matrix is manifestly of the unitary form in (9.1) with unit determinant. The Pauli matrices are, therefore, the infinitesimal generators of SU(2) and form a representation of its Lie algebra.

9.2.3 Local and Global Mappings between SU(2) and SO(3)

The matrix in (9.4) is parametrized in the same way as rotations in SO(3), namely, in terms of a rotation angle φ and a rotation axis $\boldsymbol{n}$. But, although the mapping between SU(2) and SO(3) is *locally* an isomorphism, since their algebras are isomorphic, *globally* this relationship is a homomorphism. The reason for this stems from the periodicity of the two groups: SO(3) has a periodicity of 2π, while SU(2) has a periodicity of 4π. In particular $U(0,\boldsymbol{n}) = I$, but $U(2\pi,\boldsymbol{n}) = -I$, so both of these elements are associated with the identity of SO(3). Moreover, these elements form an invariant subgroup of SU(2) (Section 2.4) which is isomorphic to the group $\mathrm{Z}_2 = \{1, -1\}$ under ordinary multiplication. In general, using the trigonometric identities,

$$
\begin{aligned}
\cos\left[\tfrac{1}{2}(\varphi + 2\pi)\right] &= -\cos\left(\tfrac{1}{2}\varphi\right) \\
\sin\left[\tfrac{1}{2}(\varphi + 2\pi)\right] &= -\sin\left(\tfrac{1}{2}\varphi\right)
\end{aligned}
$$

we find that

$$
U(\varphi + 2\pi, \boldsymbol{n}) = -U(\varphi, \boldsymbol{n})
$$

Thus, if we form the cosets of the subgroup $\{U(0,\boldsymbol{n}),U(2\pi,\boldsymbol{n})\}$, we obtain

$$\Big\{U(0,\boldsymbol{n}),U(2\pi,\boldsymbol{n})\Big\}U(\varphi,\boldsymbol{n})=\Big\{U(\varphi,\boldsymbol{n}),U(\varphi+2\pi,\boldsymbol{n})\Big\}$$

Thus, the factor group $\mathrm{SU}(2)/\mathrm{Z}_2$ is isomorphic to SO(3):

$$\mathrm{SU}(2)/\mathrm{Z}_2=\mathrm{SO}(3)$$

In fact, this double-valuedness extends to characters as well. Taking the trace of the matrix in 9.4) yields

$$2\cos\left(\tfrac{1}{2}\varphi\right)$$

If we compare this expression with that for $\chi^{(\ell)}(\varphi)$ for SO(3) with $\ell=\frac{1}{2}$, we find

$$\chi^{(1/2)}(\varphi)=\frac{\sin\varphi}{\sin\left(\frac{1}{2}\varphi\right)}=2\cos\left(\tfrac{1}{2}\varphi\right)$$

so the two-dimensional (irreducible) representation of SU(2) generated by the Pauli matrices corresponds to a representation of SO(3) with a half-integer index. The integer values of ℓ can be traced to the requirement of *single-valuedness* of the spherical harmonics, so the double-valued correspondence between SU(2) and SO(3) results in this half-integer index.

9.3 Irreducible Representations of SU(2)

When we constructed the irreducible representations of SO(2) and SO(3), we used as basis functions obtained from the coordinates $\{x,y\}$ and $\{x,y,z\}$, respectively, and to obtain higher-order irreducible representations from direct products. The basic procedure is much the same for unitary groups, except that we can no longer rely on basis states expressed in terms of coordinates. In this section, we carry out the required calculations for SU(2) and then generalize the method for SU(N) in the next section.

9.3.1 Basis States

By associating the Pauli matrices with angular momentum operators through $J_i = \frac{1}{2}\hbar\sigma_i$, we choose as our basis states the vectors

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

There are several physical interpretations of these states. For example, they can represent the two possible energy eigenstates of a spin-$\frac{1}{2}$ particle, such an electron or proton. Another possibility is that u_1 and u_2 represent the isospin eigenstates of an isospin-$\frac{1}{2}$ particle, such as a proton or a neutron. The fact that the proton and neutron are not exactly degenerate means that isospin symmetry is only an approximate symmetry. A third interpretation of u_1 and u_2 is as "up" and "down" quarks which make up nucleons. We will discuss further refinements of the quark model in the context of SU(N) later in this chapter.

9.3.2 Multiparticle Systems and Direct Products

When using basis states of SU(2) to construct multiparticle states through direct products, we must respect the indistinguishability of the particles. Thus, *measurable* properties of a quantum system cannot depend on the labelling of the particles, though wavefunctions, of course, need not obey this invariance. Consider a two-particle system, with particle '1' in state i and particle '2' in state j. The corresponding wavefunction is $\psi_{i,j}(1,2)$. We require that

$$|\psi_{i,j}(1,2)|^2 = |\psi_{i,j}(2,1)|^2$$

which implies that

$$\psi_{i,j}(2,1) = \mathrm{e}^{i\theta}\psi_{i,j}(1,2)$$

for some phase angle θ. Since a two-fold exchange restores the original labelling,

$$\psi_{i,j}(1,2) = \mathrm{e}^{i\theta}\psi_{i,j}(2,1) = \mathrm{e}^{2i\theta}\psi_{i,j}(1,2)$$

we must have that $e^{2i\theta} = 1$, or that $\theta = 0$ or $\theta = \pi$. In the first case, the wavefunction is *symmetric* under the interchange of particles,

$$\psi_{i,j}(2,1) = \psi_{i,j}(1,2)$$

while in the latter case, the wavefunction is *antisymmetric* under the interchange of particles,

$$\psi_{i,j}(2,1) = -\psi_{i,j}(1,2)$$

Consider now a two-particle system each of which occupy one of the states of SU(2). The basis of these two-particle states is comprised of $\{u_1u_1, u_1u_2, u_2u_1, u_2u_2\}$, where we have adopted the convention that the order of the states corresponds to the order of the particle coordinates, e.g., $u_1u_1 \equiv u_1(1)u_1(2)$. But not all of these states are symmetric or antisymmetric under the interchange of particles. Hence, we construct the new basis

$$\Big\{\underbrace{u_1u_1, u_1u_2 + u_2u_1, u_2u_2}_{\text{symmetric}}, \underbrace{u_1u_2 - u_2u_1}_{\text{antisymmetric}}\Big\} \tag{9.5}$$

We can compare this result with that obtained from the two-fold direct product representation of SU(2):

$$\begin{aligned}
\chi^{(1/2)}(\varphi)\chi^{(1/2)}(\varphi) &= \Big[2\cos\left(\tfrac{1}{2}\right)\Big]^2 \\
&= \Big(e^{i\varphi/2} + e^{-i\varphi/2}\Big)^2 \\
&= \Big(e^{i\varphi/2} + 1 + e^{-i\varphi/2}\Big) + 1 \\
&= \chi^{(1)}(\varphi) + \chi^{(0)}(\varphi)
\end{aligned}$$

we see that the three symmetric wavefunctions for a basis for the $\ell = 1$ irreducible representation of SO(3) and the antisymmetric wavefunctions transforms as the identical representation ($\ell = 0$) of SO(3). If we think of these as spin-$\frac{1}{2}$ particles, the symmetric state corresponds to a total spin $S = 1$, while the antisymmetric state corresponds to $S = 0$. We could proceed in this way to construct states for larger numbers of particles, but in the next section we introduce a technique which is far more efficient and which can be applied to other SU(N) groups, where the direct method described in this section becomes cumbersome.

9.3.3 Young Tableaux

Determining the dimensionalities of the irreducible representations of direct products of basis states of SU(N) is a problem which is encountered in several applications in physics and group theory. **Young tableaux** provide a diagrammatic method for carrying this out in a straightforward manner. In this section, we repeat the calculation in the preceding section to illustrate the method, and in the next section, we describe the general procedure for applying Young tableaux to SU(N).

The basic unit of a Young tableau is a 'box', shown below

$$\begin{array}{|c|}\hline \\ \hline\end{array}$$

which denotes a basis state. If there is no entry in the box, then this tableau represents any state. An entry, signified by a number denotes one of the basis states in some reference order. Thus, for SU(2), we have

$$u_1 = \boxed{1} \qquad\qquad u_2 = \boxed{2}$$

The utility of Young tableaux centers around the construction of direct products. For the two-fold direct products of SU(2) in (9.5), there are two types of states, symmetric and antisymmetric. The Young tableau for a generic two-particle symmetric state is

$$\begin{array}{|c|c|}\hline & \\ \hline\end{array}$$

and the two-particle antisymmetric state is

$$\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$$

The Young tableaux for three symmetric states in (9.5) are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline \end{array}$$

and that for the antisymmetric state is

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$$

In the framework of Young tableaux, the two-fold direct product is written as

$$\begin{array}{|c|}\hline \ \\ \hline \end{array} \times \begin{array}{|c|}\hline \ \\ \hline \end{array} = \begin{array}{|c|c|}\hline \ & \ \\ \hline \end{array} + \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array}$$

The three-fold direct product illustrates the conventions used in the construction of Young tableaux and their labelling. The generic tableaux are

$$\begin{array}{|c|}\hline \ \\ \hline \end{array} \times \begin{array}{|c|}\hline \ \\ \hline \end{array} \times \begin{array}{|c|}\hline \ \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \end{array} + \begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1} \end{array} + \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \end{array}$$

The rules for constructing the "standard" arrangement of Young tableaux are as follows

- The rows are constructed from left to right
- The columns are constructed from top to bottom
- No row is longer than any row above it
- No column is longer than any column to the left of it

Thus, with these conventions, a typical tableau is shown below:

$$\ydiagram{7,5,3,2,1}$$

The states for the three-fold direct product are as follows. There are four symmetric states:

$$\begin{ytableau} 1 & 1 & 1 \end{ytableau} \qquad \begin{ytableau} 1 & 1 & 2 \end{ytableau} \qquad \begin{ytableau} 1 & 2 & 2 \end{ytableau} \qquad \begin{ytableau} 2 & 2 & 2 \end{ytableau}$$

which correspond to a four-dimensional irreducible representation, and two "mixed" states:

$$\begin{ytableau} 1 & 1 \\ 2 \end{ytableau} \qquad \begin{ytableau} 1 & 2 \\ 2 \end{ytableau}$$

which correspond to a two-dimensional irreducible representation. There are no totally antisymmetric three-particle states because we have only two distinct basis states. Thus, the rules for entering states into Young tableaux are:

- The numbers within rows are *nondecreasing* from left to right.
- The numbers within columns are *increasing* from top to bottom.

The two sets of rules for constructing Young tableaux of generic states and identifying particular states enables the calculation of the dimensionalities in a straightforward manner, often by identifying appropriate combinatorial rules.

9.4 Young Tableaux for SU(N)

The groups SU(N) have acquired an importance in particle physics because of the quark model. This necessitates calculating direct products of basis states to determine the characteristics of particle spectra.

This, in turn, requires that we adapt the methodology of the Young tableaux developed in the preceding section to SU(N), which turns out to be straightforward given the rules stated in the preceding section. There is no change to the construction of the generic tableaux; the only changes are in the labelling of the tableaux. Consider, for example the case of a two-fold direct product of SU(3). There are six symmetric states

$$\begin{array}{|c|c|}\hline 1&1\\\hline\end{array}\qquad\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\qquad\begin{array}{|c|c|}\hline 1&3\\\hline\end{array}\qquad\begin{array}{|c|c|}\hline 2&2\\\hline\end{array}\qquad\begin{array}{|c|c|}\hline 2&3\\\hline\end{array}\qquad\begin{array}{|c|c|}\hline 3&3\\\hline\end{array}$$

and three antisymmetric states

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\qquad\qquad\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\qquad\qquad\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}$$

As is evident from these constructions, the number of states associated with a tableau of a particular topology increases sharply with the number of basis states. The rules in the preceding section allow the number of such symmetric and antisymmetric states to be calculated for SU(N). There $\frac{1}{2}N(N+1)$ symmetric states and $\frac{1}{2}N(N-1)$ antisymmetric states.

The only other modification to our discussion of SU(2) is that for larger numbers of basis states, tableaux which make no contribution to SU(2), may make a contribution to SU(N). Consider, for example, the antisymmetric three-particle state. This state vanishes for SU(2) because there are only two basis states, but for SU(3), we have

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$$

In fact, this is a direct consequence of the rule for labelling Young tableaux, and we see that, for SU(N), any column with more than N boxes makes no contribution.

9.5 Summary

In this chapter, we have extended our discussion of orthogonal groups to unitary groups. These groups play an especially important role in quantum mechanics because of their property of conserving probability density. We have constructed direct products of basis states, which are required in a number of applications of these groups. The use of Young tableaux was shown to be an especially convenient way to determine the dimensionalities of higher-dimensional irreducible representations of unitary groups and their basis functions.