1.3 *Percolation in $d = 1$ on a lattice with periodic boundary conditions.*

(i) When $s \leq L-2$, an s-cluster must be bounded by two empty sites. For $s = L-1$, there is only one empty site in the system while for $s = L$, all sites are occupied. Clearly we cannot have $s > L$. Thus

$$n(s,p) = \begin{cases} p^s(1-p)^2 & \text{for } s \leq L-2 \\ p^{L-1}(1-p) & \text{for } s = L-1 \\ p^L & \text{for } s = L \\ 0 & \text{for } s > L. \end{cases} \tag{1.3.1}$$

(ii) A cluster with $s = L$ is percolating and hence not to be characterized as being finite. Therefore, $\sum_{s=1}^{L-1} sn(s,p)$ represents the probability that a site belongs to a finite cluster.

(iii) In a $d = 1$ system of size L, the probability of an arbitrarily selected site to belong to the spanning (infinite) cluster

$$P_\infty(L,p) = p^L. \tag{1.3.2}$$

Alternatively, an occupied site either belongs to the spanning cluster or to a finite cluster ($s < L$), that is,

$$\begin{aligned} P_\infty(L,p) &= p - \sum_{s=1}^{L-1} sn(s,p) \\ &= p - (L-1)p^{L-1}(1-p) - \sum_{s=1}^{L-2} sp^s(1-p)^2 \\ &= p - (L-1)p^{L-1}(1-p) - (1-p)^2\left(p\frac{d}{dp}\right)\left(\sum_{s=1}^{L-2} p^s\right) \\ &= p - (L-1)p^{L-1}(1-p) - (1-p)^2\left(p\frac{d}{dp}\right)\left(\frac{p-p^{L-1}}{1-p}\right) \\ &= p - (L-1)p^{L-1}(1-p) - (1-p)^2 p\frac{(1-p)(1-(L-1)p^{L-2}) + (p-p^{L-1})}{(1-p)^2} \\ &= p - (L-1)p^{L-1} + (L-1)p^L - (p-p^2)(1-(L-1)p^{L-2}) - p^2 + p^L \\ &= p^L. \end{aligned} \tag{1.3.3}$$

(iv) (a) In $d = 1$ percolation,

$$\xi = -\frac{1}{\ln p} \Leftrightarrow \ln p = -\frac{1}{\xi} \Leftrightarrow p = \exp\left(-\frac{1}{\xi}\right). \quad (1.3.4)$$

Thus

$$P_\infty(L, \xi) = p^L = \left[\exp\left(-\frac{1}{\xi}\right)\right]^L = \exp\left(-\frac{L}{\xi}\right). \quad (1.3.5)$$

(b) Write the order parameter using the scaling form

$$P_\infty(\xi; L) = \exp\left(-\frac{L}{\xi}\right) = \xi^{-\beta/\nu}\mathcal{P}(L/\xi), \quad (1.3.6)$$

where

$$\beta/\nu = 0 \quad (1.3.7)$$

and a scaling function

$$\begin{aligned} \mathcal{P}(x) &= \exp(-\frac{L}{\xi}) \\ &\propto \begin{cases} \text{constant} & \text{for } L \ll \xi \\ \text{decaying rapidly} & \text{for } L \gg \xi. \end{cases} \end{aligned} \quad (1.3.8)$$