3.2 *Olami-Feder-Christensen model.*

(i) Generally we have

$$\sum_{s=1}^{\infty} s^{-a} = \begin{cases} \text{convergent} & a > 1 \\ \text{divergent} & a \leq 1 \end{cases} \tag{3.2.1}$$

Since the avalanche-size probability is normalised:

$$\sum_{s=1}^{\infty} P(s) < \infty \Rightarrow \tau_s > 1 \tag{3.2.2}$$

and the average avalanche size diverges:

$$\langle s \rangle = \sum_{s=1}^{\infty} sP(s) = \sum_{s=1}^{\infty} s^{1-\tau_s} = \infty \Rightarrow \tau_s \leq 2. \tag{3.2.3}$$

Alternatively, use the following argument

$$\sum_{s=1}^{\infty} P(s) \approx \int_1^{\infty} P(s)ds \propto [s^{1-\tau_s}]_1^{\infty} \tag{3.2.4}$$

which is only convergent in the upper limit for $\tau_s > 1$ and

$$\langle s \rangle = \sum_{s=1}^{\infty} sP(s) \approx \int_1^{\infty} sP(s)ds \propto [s^{2-\tau_s}]_1^{\infty} \tag{3.2.5}$$

which is only divergent in the upper limit for $\tau_s \leq 2$ (logarithmically so for $\tau_s = 2$).

(ii) As the cutoff event size s_ξ diverges for $\alpha \to \alpha_c$,the limiting function of $P(s)$ will be a pure power law, that is,

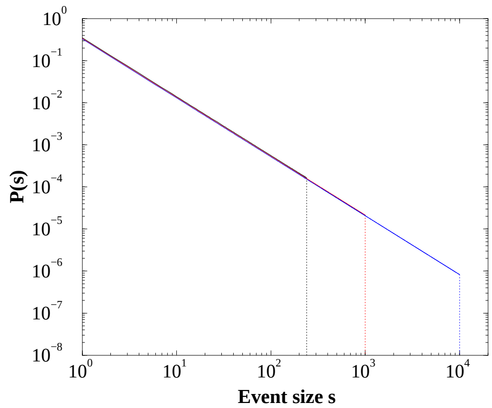
$$P(s) = \begin{cases} s^{-\tau_s} & \text{for } s \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

(iii)

$$\begin{aligned} \langle s^k \rangle = \sum_{s=1}^{\infty} s^k P(s) \approx \int_1^{\infty} s^k P(s)ds = \int_1^{s_\xi} s^{k-\tau_s} ds \\ \propto s_\xi^{1+k-\tau_s} \propto (\alpha_c - \alpha)^{\frac{\tau_s - k - 1}{\sigma}} \end{aligned}$$

that is,

$$\gamma_k = \frac{\tau_s - k - 1}{\sigma}.$$



(iv) The dynamical rules of the model are motivated by the dynamics of earthquakes in which there are two separate time scales. One is defined by the motion of the tectonic plates, and the other is the duration of an earthquake. The former time scale is much larger than the latter. We separate the time scales by considering the earthquake as instantaneous, that is, the system is not driven during an earthquake.
The algorithm for the system is as following:

- Define random initial strains in the system.
- Strain is accumulated uniformly across the system as the rigid plates move.
- When the strain in a certain site is above the threshold value F_{th} this site will relax according to the equation

$$\begin{aligned} F_{nn} &\to F_{nn} + \alpha F_{ij} \, , \\ F_{ij} &\to 0, \end{aligned} \tag{3.2.6}$$

  where F_{nn} denote the nearest-neighbour blocks of the relaxing block (i, j) and $\alpha = K/(4K + K_L)$.
  This may cause neighbouring sites to exceed the threshold value, in which case these sites relax simultaneously, and so on. The triggered earthquake will stop when there are no sites left with a strain above the threshold value.
- Strain starts to accumulate once again.

As the relaxation dissipates F_{ij} but an amount of $4\alpha F_{ij}$ is redistributed, we refer to 4α as the level of conservation.

(v) (a) The model is considered to be critical if, for a given value of α, the event size distribution $P(s)$ is a power law with a cutoff size that diverges with systems size L. This will also imply that the average event size will diverge with system size. If, on the other hand, the cutoff size does not increase with system size, the model would be non-critical.
Clearly, for $\alpha = 0$, the blocks do not interact at all, and all the avalanches are of size 1, that is, the model is non-critical. For $\alpha = 0.25$, the model is conservative (conservation level = 1), and all the dissipation will take place at the boundary only. Thus one would expect the average avalanche size to diverge with system size, consistent with a power law distribution $P(s)$.
As the model is non-critical for $\alpha = 0$ and critical for $\alpha = 0.25$ there must be a crossover at some critical value α_c from a critical to a non-critical behaviour as α decreases from 0.25 to 0. Where the transition happens is still an unsettled question. There are claims that $\alpha_c = 0.25$ and $\alpha_c = 0$.

3.3 *Modified Bak-Tang-Wiesenfeld model on a tree-like lattice.*

(i) (a) Each of the N sites can be in one of h_c state, $h_i = 0, 1, \ldots, h_c - 1$. Thus there are a total of h_c^N stable configurations.
(b) Stable configurations are either transient or recurrent configurations. Transient configurations are not encountered once the set of recurrent configurations is reached. The set of recurrent configurations is commonly known as the attractor of the dynamics.
(c) Given a configuration in the set of recurrent states. Simply by adding $h_c - 1 - h_i$ grains to each of the i sites we recover the minimally stable configuration with $h_i = h_c - 1$ for all sites i.
Adding one grain at the root of the tree-like structure in the minimally stable configuration will induce an avalanches in which all the grains will leave the system at the boundary and leave the system empty.
Since the empty configuration is a recurrent state, all stable configurations will be recurrent as they can be reached from the empty configuration by adding grains in a pre-

scribed way.

(ii) (a) A site with $h = h_c - 1$ will topple if it receives one grain. Such sites occur with probability P_{h_c-1}. Sites with $h < h_c - 1$ will not topple upon receiving one grain. Such sites occur with probability $1 - P_{h_c-1}$. Since a toppling site adds one grain to its h_c downwards neighbours the probability of causing b new sites to topple is determined by the binomial distribution

$$p_b = \binom{h_c}{b} P_{h_c-1}^b \left(1 - P_{h_c-1}\right)^{h_c-b} \quad b = 0, \ldots, h_c.$$

(b) The number of trials are h_c, each with a probability P_{h_c-1} of causing a new toppling. Therefore, the average number of new topplings

$$\langle b \rangle = \sum_{b=0}^{h_c} b p_b = h_c P_{h_c-1}.$$

(iii) Since the probability P_h must be normalised,

$$\sum_{h=0}^{h_c-1} P_h = h_c P_h = 1 \Leftrightarrow P_h = \frac{1}{h_c}.$$

Therefore, clearly

$$\langle b \rangle = h_c P_{h_c-1} = 1.$$

This is the critical branching ratio for a branching process. Thus the model self-organised into a critical state in which there are avalanches of all sizes, limited by the system size only.

(iv) (a) In a tree with $h_c = 2$ in a stable configuration, each site can be in one of two states, either $h_i = 0$ or $h_i = 1$. Define for now sites with $h_i = 1$ as occupied sites and sites with $h_i = 0$ as empty sites. Then the probability that a site is occupied is $P_{h=1} = 1/h_c = 1/2$, the critical occupation probability of percolation model on a Bethe lattice with $z = 3$. However, the sandpile model organises itself to this critical state.

(b) When adding a grain to an arbitrary site, it topples with probability P_{h_c-1}. Define B to be the contribution to the average avalanche size from a given sub-branch. Then the average avalanche size is

$$\langle s \rangle = P_{h_c-1}\left(1 + h_c B\right), \qquad (3.3.1)$$

where the first term is the contribution from the toppling site itself and the second term is the contribution from the h_c sub-branches. If the parent of a sub-branch has $h_i < h_c - 1$ there is no contribution. If, however, the parent of a sub-branch has $h_i = h_c - 1$, that parent contributes its own toppling together with a contribution B from each of its h_c subbranches. The contribution from a subbranch is identical to the contribution from a branch because all sites are equivalent. Thus

$$B = 0 \times (1 - P_{h_c-1}) + [1 + h_c B] \times P_{h_c-1}$$

from which

$$B = \frac{P_{h_c-1}}{1 - h_c P_{h_c-1}}.$$

Substituting this result into Equation (3.3.1) we find

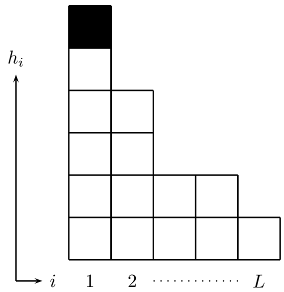
$$\langle s \rangle = P_{h_c-1}\left(1 + h_c \frac{P_{h_c-1}}{1 - h_c P_{h_c-1}}\right) = \frac{P_{h_c-1}}{1 - h_c P_{h_c-1}}$$

which diverges for $P_{h_c-1} \to 1/h_c$.

3.4 *Oslo model and moments.*

(i) Starting from an empty system, a pile will gradually form when adding grains. However, eventually, after a transient period, the pile will cease to grow and, in average, the number of grains added at the left boundary will leave the system at the right boundary. Once the system has reached the attractor of the dynamics, the avalanches initiated by adding grains at the left boundary is only limited by the size of the system. The system has, by itself, organised into a state in which the average avalanche scales with system size, the signature of criticality.

(ii) (a) Define the local slope $z_i = h_i - h_{i+1}, i = 1, \dots L$ with $h_{L+1} = 0$. In the one-dimensional Oslo model, the critical slopes, $z_i^c(t)$, dependent on position and time.



The algorithm for the dynamics is defined as follows.

1. Place the pile in an arbitrary stable configuration with $z_i \le z_i^c$ for all i.
2. Add a grain at site $i = 1$, that is, $z_1 \to z_1 + 1$.
3. If $z_i > z_i^c$, the site relaxes and

$$\begin{aligned} z_i &\to z_i - 2 \\ z_{i\pm1} &\to z_{i\pm1} + 1 \end{aligned}$$

except when boundary sites topple, where, respectively,

$$\begin{aligned} z_1 &\to z_1 - 2 & z_L \quad &\to z_L - 1 \\ z_2 &\to z_2 + 1 \quad \text{for } i = 1 & z_{L-1} &\to z_{L-1} + 1 \quad \text{for } i = L. \end{aligned}$$

Chose a new critical slope z_i^c at toppling site. A stable configuration is reached when $z_i \le z_i^c$ for all i.
4. Proceed to step **2.** and reiterate.

(b) The pile will eventually reach a statistically stationary state where, on average, the number of grains added will leave the system at the open boundary. Configurations are either transient configuration or recurrent configurations. Recurrent configurations will appear again and again if we wait long enough.

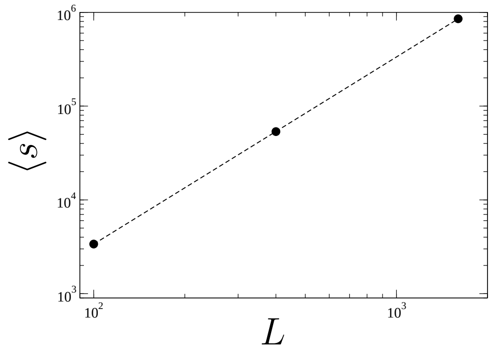
(iii) The kth moment

$$\begin{aligned}\langle s^k \rangle &= \sum_{s=1}^{\infty} s^k P(s, L) \\ &= \sum_{s=1}^{\infty} s^{k-\tau_s} \mathcal{G}(s/L^D) \\ &\approx \int_1^{\infty} s^{k-\tau_s} \mathcal{G}(s/L^D) ds \\ &= \int_{1/L^D}^{\infty} (uL^D)^{k-\tau_s} \mathcal{G}(u) L^D du \quad \text{with } u = s/L^D \\ &= L^{D(k+1-\tau_s)} \int_{1/L^D}^{\infty} u^{k-\tau_s} \mathcal{G}(u) du\end{aligned}$$

For $L \gg 1$, the lower limit of the integral approaches zero, and the integral becomes just a numerical factor. Therefore,

$$\begin{aligned}\langle s^k \rangle &\approx L^{D(k+1-\tau_s)} \int_0^{\infty} u^{k-\tau_s} \mathcal{G}(u) du \quad \text{for } L \gg 1 \\ &\propto L^{D(k+1-\tau_s)}.\end{aligned}$$

(iv) (a) Plotting $\log\langle s \rangle$ versus $\log L$, we see that the data fall on a line with slope approximately 2.



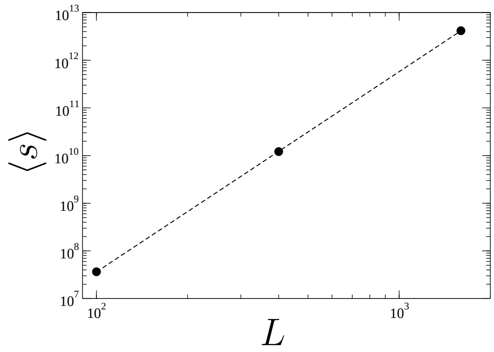
Therefore

$$\langle s \rangle \propto L^2 \propto L^{D(2-\tau_s)} \quad \text{for } L \gg 1, \tag{3.4.1}$$

implying the scaling relation

$$D(2 - \tau_s) = 2. \tag{3.4.2}$$

(b) Plotting for example $\log\langle s^2\rangle$ versus $\log L$, the data fall on a line with slope approximately 4.2.



Thus

$$D(2-\tau_s) = 2 \qquad (3.4.3a)$$
$$D(3-\tau_s) = 4.2 \qquad (3.4.3b)$$

from which, by subtraction

$$D \approx 2.2 \qquad (3.4.4)$$

and using the scaling relation in Equation (3.4.2)

$$\tau_s = 2 - 2/D \approx 1.1. \qquad (3.4.5)$$

3.5 *Moment ratios and universality.*

(i) Given that the avalanche-size probability

$$P(s;L) = as^{-\tau_s}\mathcal{G}\left(s/bL^D\right) \quad \text{for } s \gg 1, L \gg 1$$

then be rearranging we find

$$\frac{1}{a}s^{\tau_s}P(s;L) = \mathcal{G}\left(s/bL^D\right) \quad \text{for } s \gg 1, L \gg 1.$$

For a given system a and b are constant. The L.H.S. is a function of two variables s and L while the R.H.S. is a function of one variable only, s/bL^D. Hence by plotting the transformed avalanche-size probabilities $\frac{1}{a}s^{\tau_s}P(s;L)$ versus the rescaled avalanche size, s/bL^D, the data should, for $s \gg 1$ collapse onto the graph for the scaling function $\mathcal{G}$.

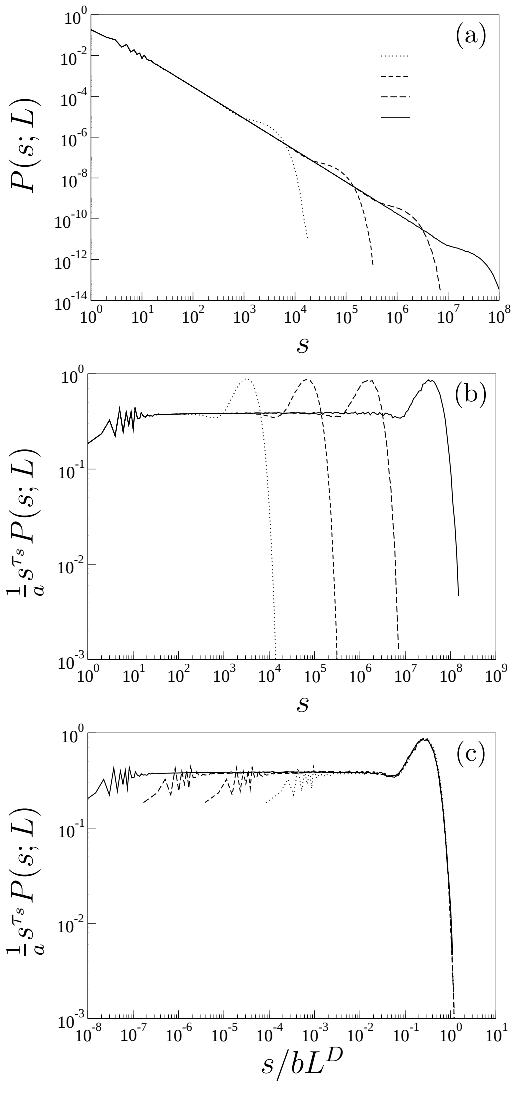


Fig. 3.5.1 (a) The avalanche-size probabilities, $P(s;L)$, versus avalanche size, s. The four curves correspond to lattices of increasing size marked with lines of increasing dash length. (b) The transformed avalanche-size probabilities, $\frac{1}{a}s^{\tau_s}P(s;L)$, versus avalanche size, s. (b) Plotting the transformed avalanche-size probability, $\frac{1}{a}s^{\tau_s}P(s;L)$, versus the rescaled avalanche size, s/bL^D, produces a data collapse onto a universal scaling function $\mathcal{G}$ when using the appropriate exponents D and τ_s.

(ii) (a) Assuming the scaling form of the avalanche-size probability is valid for all s and converting the sum into an integral we find

$$\begin{aligned}\langle s^k\rangle &= \sum_{s=1}^{\infty} s^k P(s;L)\\ &= \sum_{s=1}^{\infty} a s^{k-\tau_s}\mathcal{G}\left(s/bL^D\right)\\ &\approx \int_1^{\infty} a s^{k-\tau_s}\mathcal{G}\left(s/bL^D\right)\, ds\\ &= \int_{1/bL^D}^{\infty} a(ubL^D)^{k-\tau_s}\mathcal{G}\left(u\right)\, bL^D du \quad \text{with } u = s/bL^D\\ &= a(bL^D)^{1+k-\tau_s}\int_{1/bL^D}^{\infty} u^{k-\tau_s}\mathcal{G}\left(u\right)\, du\\ &= L^{D(1+k-\tau_s)} a b^{1+k-\tau_s}\int_0^{\infty} u^{k-\tau_s}\mathcal{G}\left(u\right)\, du,\end{aligned}$$

since the lower limit of the integral tends to zero as $L \to \infty$. Hence we identify the universal exponent and the non-universal amplitude

$$\begin{aligned}\gamma_k &= D(1+k-\tau_s) && \text{universal}\\ \Gamma_k &= a b^{1+k-\tau_s}\int_0^{\infty} u^{k-\tau_s}\mathcal{G}\left(u\right)\, du && \text{non-universal.}\end{aligned}$$

(b) The moment ratio

$$g_k = \frac{\langle s^k\rangle\langle s\rangle^{k-2}}{\langle s^2\rangle^{k-1}} = \frac{\Gamma_k L^{D(1+k-\tau_s)}(\Gamma_1 L^{D(2-\tau_s)})^{k-2}}{(\Gamma_2 L^{D(3-\tau_s)})^{k-1}} = \frac{\Gamma_k \Gamma_1^{k-2}}{\Gamma_2^{k-1}}$$

which is clearly independent of the non-universal constants a and b.

(iii) (a) In the derivation above, we assumed the scaling form of the avalanche-size probability. However, that is only valid for $L \gg 1$. Hence only for $L \to \infty$ will the moment ratio g_k be independent on system size.

(b) Model A and B might be in the same universality class. However, Model C must belong to another universality class. Otherwise the asymptotic value of g_3 cannot be different.