Answers to exercises

1.1 *Moments and moment ratio of the cluster number density in* $d = 1$*.*

(i) In $d = 1$, the cluster number density $n(s,p) = (1-p)^2 p^s$. Thus the kth moment M_k of the cluster size density

$$\begin{aligned}
M_k &= \sum_{s=1}^{\infty} s^k n(s,p) \\
&= \sum_{s=1}^{\infty} s^k (1-p)^2 p^s \\
&= (1-p)^2 \sum_{s=1}^{\infty} s^k p^s \\
&= (1-p)^2 \sum_{s=1}^{\infty} s^k \exp[s\ln(p)] \\
&= (1-p)^2 \sum_{s=1}^{\infty} s^k \exp(-s/s_\xi), \quad \text{with } s_\xi = -\frac{1}{\ln(p)} \\
&\approx (1-p)^2 \int_1^{\infty} s^k \exp(-s/s_\xi)\, ds, \qquad u = s/s_\xi; du = ds/s_\xi \\
&= (1-p)^2 \int_{1/s_\xi}^{\infty} (u s_\xi)^k \exp(-u)\, s_\xi du \\
&= (1-p)^2 {s_\xi}^{k+1} \int_{1/s_\xi}^{\infty} u^k \exp(-u)\, du \\
&= (1-p)^2 \left(\frac{-1}{\ln(p)}\right)^{k+1} \int_{-\ln(p)}^{\infty} u^k \exp(-u)\, du. \qquad (1.1.1)
\end{aligned}$$

Letting $p \to p_c^-$, the lower limit of the integral tends to zero (as $p_c = 1$), and the integral becomes the integral representation of the Gamma function. Using the Taylor expansion $\ln(p) = \ln[1-(1-p)] \approx -(1-p)$ for $p \to p_c^-$ we find

$$\begin{aligned}
M_k &= (1-p)^2 \frac{1}{(1-p)^{k+1}} k! \\
&= k!(p_c - p)^{1-k} \qquad (1.1.2)
\end{aligned}$$

so we identity $\Gamma_k = k!$ and $\gamma_k = k-1$.

Alternative derivation with the use of "a trick":

$$\begin{aligned}
M_k &= \sum_{s=1}^{\infty} s^k n(s,p) \\
&= (1-p)^2 \sum_{s=1}^{\infty} \left(p\frac{d}{dp}\right)^k p^s \quad \text{the "trick"} \\
&= (1-p)^2 \left(p\frac{d}{dp}\right)^k \sum_{s=1}^{\infty} p^s \\
&= (1-p)^2 \left(p\frac{d}{dp}\right)^k \frac{p}{1-p} \\
&\stackrel{?}{=} k!(1-p)^{1-k} \qquad \text{for } k \geq 2,
\end{aligned} \tag{1.1.3}$$

followed by proof by induction.
First the case $k = 2$.

$$\begin{aligned}
M_{k=2} &= (1-p)^2 \left(p\frac{d}{dp}\right)^2 \frac{p}{1-p} \\
&= (1-p)^2 \left(p\frac{d}{dp}\right) p\frac{(1-p)\cdot 1 + p}{(1-p)^2} \\
&= (1-p)^2 \left(p\frac{d}{dp}\right) \frac{p}{(1-p)^2} \\
&= (1-p)^2 p\frac{(1-p)^2 \cdot 1 + p \cdot 2(1-p)}{(1-p)^4} \\
&= p\frac{(1-p) + 2p}{1-p} \\
&= \frac{p+p^2}{1-p} \\
&\to \frac{2}{1-p} \\
&= 2!(p_c - p)^{-1} \qquad \text{for } p \to p_c = 1.
\end{aligned} \tag{1.1.4}$$

Now, assume that

$$M_k = (1-p)^2 \left(p\frac{d}{dp}\right)^k \frac{p}{1-p} = k!(1-p)^{1-k} \qquad \text{for } k \geq 2. \tag{1.1.5}$$

Then

$$\begin{aligned}
M_{k+1} &= (1-p)^2 \left(p\frac{d}{dp}\right)^{k+1} \frac{p}{1-p} \\
&= (1-p)^2 \left(p\frac{d}{dp}\right)\left(p\frac{d}{dp}\right)^{k} \frac{p}{1-p} \\
&= (1-p)^2 \left(p\frac{d}{dp}\right) k!(1-p)^{-1-k} \quad \text{using the assumption Equation (1.1.5)} \\
&= (1-p)^2 pk!(-1-k)(1-p)^{-2-k}\cdot(-1) \\
&= p(k+1)!(1-p)^{-k} \\
&\to (k+1)!(1-p)^{1-(k+1)} \quad \text{for } p\to p_c=1, \qquad (1.1.6)
\end{aligned}$$

so the assumption Equation (1.1.5) is true for $k+1$, which completes our proof.

(ii) Note that $M_1 = \sum_{s=1}^{\infty} sn(s,p) = p$ for $p<1$ so $\Gamma_1 = p$ and $\gamma_1 = 0$. Hence,the moment ratio

$$\begin{aligned}
g_k &= \frac{M_k M_1^{k-2}}{M_2^{k-1}} \\
&= \frac{\Gamma_k (1-p)^{1-k}\Gamma_1^{k-2}\left[(1-p)^0\right]^{1-k}}{\Gamma_2^{k-1}\left[(1-p)^{-1}\right]^{k-1}} \\
&= \frac{\Gamma_k \Gamma_1^{k-2}}{\Gamma_2^{k-1}} \qquad (1.1.7)
\end{aligned}$$

Since $\Gamma_1 \to 1$ for $p \to p_c^-$ we find

$$\begin{aligned}
g_k &\to \frac{\Gamma_k}{\Gamma_2^{k-1}} \quad \text{for } p\to p_c^- \\
&= \frac{k!}{2!^{k-1}} \qquad (1.1.8)
\end{aligned}$$

which is a constant for a given k.

1.2 *Site percolation and site-bond percolation in $d=1$.*

(i) (a) A percolating (infinite) cluster is present at p_c. In one dimension, a percolating cluster can have no empty sites. Therefore $p_c = 1$.

(b) A cluster of size s has s consecutive sites occupied, each with probability p, and two empty sites, one at either end, each with probability $(1-p)$, so

$$n(s,p) = p^s(1-p)^2.$$

(c) Since $n(s,p)$ is the number of s clusters per lattice site, $sn(s,p)$ is the probability that an arbitrary site belongs to an s cluster. Summing over all possible sizes of clusters, we obtain the probability that an arbitrary site is occupied, that is,

$$\sum_{s=1}^{\infty} sn(s,p) = p \quad \text{for } p < 1.$$

This identity is not valid at $p = 1$ where the percolating cluster is occupying all the lattice leaving $n(s,p) = 0$ for $p = 1$.

(d) We find that

$$\begin{aligned}
\sum_{s=1}^{\infty} s^2 p^s &= \sum_{s=1}^{\infty} \left(p\frac{d}{dp}\right)\left(p\frac{d}{dp}\right) p^s \\
&= \left(p\frac{d}{dp}\right)\left(p\frac{d}{dp}\right) \sum_{s=1}^{\infty} p^s \\
&= \left(p\frac{d}{dp}\right)\left(p\frac{d}{dp}\right) \frac{p}{1-p} \quad && \text{for } p < 1 \\
&= \left(p\frac{d}{dp}\right) p\frac{(1-p)+p}{(1-p)^2} \quad && \text{for } p < 1 \\
&= \left(p\frac{d}{dp}\right) \frac{p}{(1-p)^2} \quad && \text{for } p < 1 \\
&= p\frac{(1-p)^2 + p2(1-p)}{(1-p)^4} \quad && \text{for } p < 1 \\
&= p\frac{1+p}{(1-p)^3} \quad && \text{for } p < 1.
\end{aligned}$$

(e) Using the above results we find

$$\begin{aligned}\chi(p) &= \frac{\sum_{s=1}^{\infty} s^2 p^s (1-p)^2}{\sum_{s=1}^{\infty} sn(s,p)} \\ &= \frac{(1-p)^2 p \frac{1+p}{(1-p)^3}}{p} \\ &= \frac{1+p}{1-p}.\end{aligned}$$

(f) Therefore

$$\chi(p) \to \frac{1+p_c}{1-p} = \frac{2}{p_c - p} = \quad \text{for } p \to p_c^-,$$

so we identify the amplitude $\Gamma = 2$ and the critical exponent $\gamma = 1$.

(ii) (a) A percolating (infinite) cluster is present at (p_c, q_c). Therefore, no sites nor bonds can be empty, implying $(p_c, q_c) = (1,1)$.

(b) An s cluster has s consecutive site occupied, each with probability p, and $s-1$ consecutive bonds occupied, each with probability q. Since pq is the probability to have a site-bond occupied, $(1-pq)^2$ is the probability that a cluster does not continue at either end. Therefore

$$n(s,p,q) = p^s q^{s-1} (1-pq)^2 .$$

(c) First,

$$\begin{aligned}\sum_{s=1}^{\infty} sn(s,p,q) &= \sum_{s=1}^{\infty} sp^s q^{s-1} (1-pq)^2 \\ &= \frac{1}{q} \sum_{s=1}^{\infty} s(pq)^s (1-pq)^2 \\ &= \frac{1}{q} pq \\ &= p\end{aligned}$$

and similarly

$$\begin{aligned}\sum_{s=1}^{\infty} s^2 n(s,p,q) &= \sum_{s=1}^{\infty} s^2 p^s q^{s-1} \left(1-pq\right)^2 \\ &= \frac{1}{q}\left(1-pq\right)^2 \sum_{s=1}^{\infty} s^2 (pq)^s \\ &= \frac{1}{q}\left(1-pq\right)^2 pq \frac{1+pq}{(1-pq)^3} \\ &= p\frac{1+pq}{1-pq}\end{aligned}$$

so that

$$\chi(p,q) = \frac{1+pq}{1-pq}.$$

This result is identical to that of site percolation if we identify the occupation probability with pq, that is, a site-bond is the equivalent of a site.