1.3 *Percolation in $d = 1$ on a lattice with periodic boundary conditions.*

(i) When $s \leq L-2$, an s-cluster must be bounded by two empty sites. For $s = L-1$, there is only one empty site in the system while for $s = L$, all sites are occupied. Clearly we cannot have $s > L$. Thus

$$n(s,p) = \begin{cases} p^s(1-p)^2 & \text{for } s \leq L-2 \\ p^{L-1}(1-p) & \text{for } s = L-1 \\ p^L & \text{for } s = L \\ 0 & \text{for } s > L. \end{cases} \tag{1.3.1}$$

(ii) A cluster with $s = L$ is percolating and hence not to be characterized as being finite. Therefore, $\sum_{s=1}^{L-1} sn(s,p)$ represents the probability that a site belongs to a finite cluster.

(iii) In a $d = 1$ system of size L, the probability of an arbitrarily selected site to belong to the spanning (infinite) cluster

$$P_\infty(L,p) = p^L. \tag{1.3.2}$$

Alternatively, an occupied site either belongs to the spanning cluster or to a finite cluster ($s < L$), that is,

$$\begin{aligned} P_\infty(L,p) &= p - \sum_{s=1}^{L-1} sn(s,p) \\ &= p - (L-1)p^{L-1}(1-p) - \sum_{s=1}^{L-2} sp^s(1-p)^2 \\ &= p - (L-1)p^{L-1}(1-p) - (1-p)^2 \left(p\frac{d}{dp}\right)\left(\sum_{s=1}^{L-2} p^s\right) \\ &= p - (L-1)p^{L-1}(1-p) - (1-p)^2 \left(p\frac{d}{dp}\right)\left(\frac{p-p^{L-1}}{1-p}\right) \\ &= p - (L-1)p^{L-1}(1-p) - (1-p)^2 p\frac{(1-p)(1-(L-1)p^{L-2}) + (p-p^{L-1})}{(1-p)^2} \\ &= p - (L-1)p^{L-1} + (L-1)p^L - (p-p^2)(1-(L-1)p^{L-2}) - p^2 + p^L \\ &= p^L. \end{aligned} \tag{1.3.3}$$

(iv) (a) In $d = 1$ percolation,

$$\xi = -\frac{1}{\ln p} \Leftrightarrow \ln p = -\frac{1}{\xi} \Leftrightarrow p = \exp\left(-\frac{1}{\xi}\right). \qquad (1.3.4)$$

Thus

$$P_\infty(L,\xi) = p^L = \left[\exp\left(-\frac{1}{\xi}\right)\right]^L = \exp\left(-\frac{L}{\xi}\right). \qquad (1.3.5)$$

(b) Write the order parameter using the scaling form

$$P_\infty(\xi; L) = \exp\left(-\frac{L}{\xi}\right) = \xi^{-\beta/\nu}\mathcal{P}(L/\xi), \qquad (1.3.6)$$

where

$$\beta/\nu = 0 \qquad (1.3.7)$$

and a scaling function

$$\mathcal{P}(x) = \exp(-\frac{L}{\xi}) \propto \begin{cases} \text{constant} & \text{for } L \ll \xi \\ \text{decaying rapidly} & \text{for } L \gg \xi. \end{cases} \qquad (1.3.8)$$

1.4 *Cluster number density scaling functions in d=1 and the Bethe lattice.*

(i) (a) Rewriting the cluster number density in $d = 1$ we find

$$\begin{aligned} n(s,p) &= (1-p)^2 p^s \\ &= (p_c - p)^2 \exp(-s/s_\xi) \qquad \text{with } s_\xi = -\frac{1}{\ln p} \\ &= s^{-2}[s(p_c-p)]^2 \exp(-s/s_\xi) \\ &\approx s^{-2}\,(s/s_\xi)^2 \exp(-s/s_\xi) \qquad \text{for } p \to p_c^- \\ &= s^{-2}\mathcal{G}_{1d}(s/s_\xi) \end{aligned} \tag{1.4.1}$$

with

$$\mathcal{G}_{1d}(s/s_\xi) = (s/s_\xi)^2 \exp(-s/s_\xi). \tag{1.4.2}$$

and

$$s_\xi \to (p_c - p)^{-1} \quad \text{for } p \to p_c^-. \tag{1.4.3}$$

Thus we identify

$$\tau = 2, \tag{1.4.4a}$$
$$\sigma = 1, \tag{1.4.4b}$$
$$a = 1, \tag{1.4.4c}$$
$$b = 1. \tag{1.4.4d}$$

(b) From the graph of the scaling function $\mathcal{G}_{1d}$, see Figure 1.4.1, we see that for small arguments $s \ll s_\xi$, the function increases quadratically in the argument s/s_ξ while it decays exponentially fast for $s \gg s_\xi$. Indeed, such cluster sizes are exponentially rare as the characteristic cluster size s_ξ is the typical size of the largest cluster.

(c) The scaling function $\mathcal{G}_{1d}(x) = x^2 \exp(-x)$ and

$$\mathcal{G}_{1d}^{(1)}(x) = 2x\exp(-x) - x^2\exp(-x) = (2x - x^2)\exp(-x)$$
$$\mathcal{G}_{1d}^{(2)}(x) = (2 - 2x - 2x + x^2)\exp(-x) = (2 - 4x + x^2)\exp(-x)$$

Hence $\mathcal{G}_{1d}(0) = \mathcal{G}_{1d}^{(1)}(0) = 0, \mathcal{G}_{1d}^{(2)}(0) = 2$. Thus the Taylor

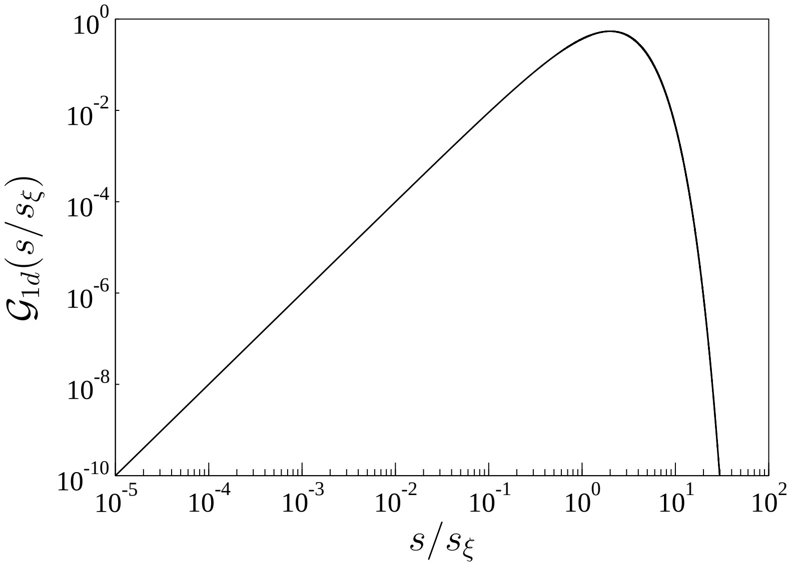

Fig. 1.4.1 The scaling function $\mathcal{G}_{1d}$ in $d = 1$ increases like $(s/s_\xi)^2$ for small arguments and decays (exponentially) fast for large arguments.

expansion of $\mathcal{G}_{1d}$ around zero,

$$\begin{aligned}\mathcal{G}_{1d}(s/s_\xi) &= \mathcal{G}_{1d}(0) + \mathcal{G}_{1d}^{(1)}(0)s/s_\xi + \frac{1}{2}\mathcal{G}_{1d}^{(2)}(0)\,(s/s_\xi)^2 + \cdots \\ &= (s/s_\xi)^2 + \cdots \end{aligned} \tag{1.4.5}$$

which is consistent with Figure 1.4.1.

(ii) (a) On a Bethe lattice with $z = 3$ where $p_c = 1/2$ we have

$$n(s,p) \propto s^{-5/2} \exp(-s/s_\xi) \qquad s \gg 1$$
$$s_\xi = -\frac{1}{\ln(4p-4p^2)} \to \frac{1}{4}\,(p-p_c)^{-2} \quad \text{for } p \to p_c.$$

Thus we identify the scaling function

$$\mathcal{G}_{\mathrm{Bethe}}(s/s_\xi) = \exp(-s/s_\xi). \tag{1.4.6}$$

with

$$\tau = 5/2 \tag{1.4.7a}$$
$$\sigma = 1/2 \tag{1.4.7b}$$
$$b = 1/4. \tag{1.4.7c}$$

It would be possible to determine a by applying a normalisation constraint. For example when $p < p_c$ the cluster number density must satisfy

$$\sum_{s=1}^{\infty} sn(s,p) = a \sum_{s=1}^{\infty} s^{1-\tau} \mathcal{G}_{\mathrm{Bethe}}(s/s_\xi) = p. \tag{1.4.8}$$

This constraint will determine a.

(b) From the graph of the scaling function $\mathcal{G}_{\mathrm{Bethe}}$, see Figure 1.4.2, we see that for small arguments $s \ll s_\xi$, the function is approximately constant while it decays exponentially fast for $s \gg s_\xi$. Indeed, such cluster sizes are exponentially rare as the characteristic cluster size s_ξ is the typical size of the largest cluster.

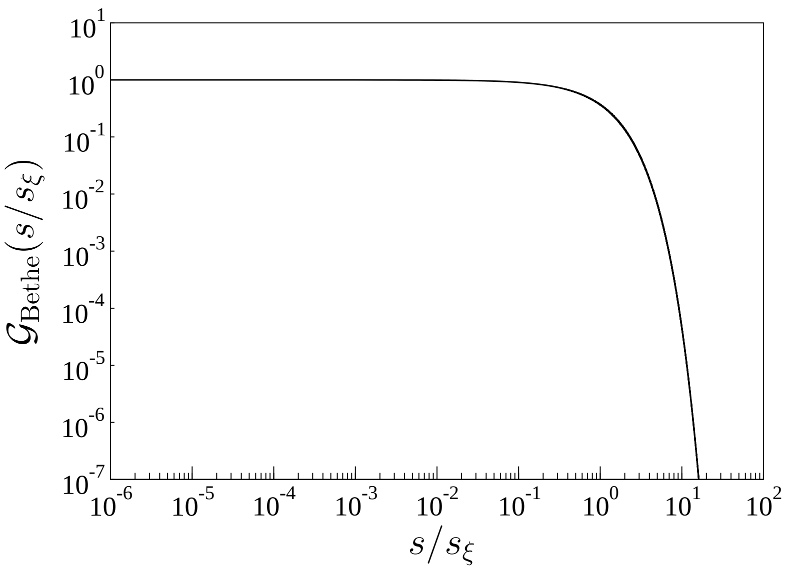

Fig. 1.4.2 The scaling function $\mathcal{G}_{\mathrm{Bethe}}$ for the Bethe lattice is approximately constant for small arguments and decays exponentially fast for large arguments.

(c) Clearly

$$\mathcal{G}_{\mathrm{Bethe}}(x) = 1 - x + \cdots \approx 1, \tag{1.4.9}$$

consistent with Figure 1.4.2.

1.5 *Moments of the cluster number density.*

(i) We approximate the sum by an integral:

$$\begin{aligned}
M_k(p) &= \sum_{s=1}^{\infty} s^k n(s,p) \\
&= \sum_{s=1}^{\infty} a s^{k-\tau} \mathcal{G}(s/s_\xi) \\
&\approx \int_1^{\infty} a s^{k-\tau} \mathcal{G}(s/s_\xi)\, ds \\
&= \int_{1/s_\xi}^{\infty} a\, (s_\xi u)^{k-\tau}\, \mathcal{G}(u) s_\xi\, du \qquad \text{with } u = s/s_\xi \\
&= s_\xi^{k+1-\tau} a \int_{1/s_\xi}^{\infty} u^{k-\tau} \mathcal{G}(u)\, du \\
&= |p-p_c|^{-(k+1-\tau)/\sigma} a b^{k+1-\tau} \int_0^{\infty} u^{k-\tau} \mathcal{G}(u)\, du \quad \text{for } p \to p_c \\
&= \Gamma_k\, |p-p_c|^{-\gamma_k} \qquad (1.5.1)
\end{aligned}$$

where

$$\gamma_k = \frac{k+1-\tau}{\sigma} \qquad (1.5.2a)$$

$$\Gamma_k = a b^{k+1-\tau} \int_0^{\infty} u^{k-\tau} \mathcal{G}(u)\, du. \qquad (1.5.2b)$$

The critical amplitude Γ_k is just a number independent of p. Note that we recover the scaling relation

$$\gamma = \frac{3-\tau}{\sigma} \qquad (1.5.3)$$

by letting $k = 2$.

(ii) The moment ratio

$$\begin{aligned}
g_k &= \frac{M_k M_1^{k-2}}{M_2^{k-1}} \\
&= \frac{\Gamma_k \Gamma_1^{k-2}}{\Gamma_2^{k-1}} \qquad (1.5.4) \\
&= \frac{\int_0^{\infty} u^{k-\tau} \mathcal{G}(u)\, du \left[\int_0^{\infty} u^{1-\tau} \mathcal{G}(u)\, du\right]^{k-2}}{\left[\int_0^{\infty} u^{2-\tau} \mathcal{G}(u)\, du\right]^{k-1}}
\end{aligned}$$

(iii) In $d = 1$ percolation, $\tau = 2, \sigma = 1, a = 1, b = 1$ and the scaling function $\mathcal{G}_{1d}(u) = u^2 \exp(-u)$ so

$$\begin{aligned}\Gamma_k &= \int_0^\infty u^k \exp(-u)\, du \\ &= k!\end{aligned}$$

1.6 *Universality of the ratio of amplitudes for the average cluster size.* By definition

$$\chi(p) = \frac{\sum_{s=1}^\infty s^2 n(s,p)}{\sum_{s=1}^\infty s n(s,p)} \tag{1.6.1}$$

where the denominator $\sum_{s=1}^\infty sn(s,p) = p_c$ at $p = p_c$. Since we are ultimately interested in the limit $p \to p_c$, we simply substitute the denominator with p_c.
We thus find

$$\begin{aligned}p_c\chi(p) &= \sum_{s=1}^\infty s^2 n(s,p) \\ &= \sum_{s=1}^\infty a s^{2-\tau} \mathcal{G}_\pm(s/s_\xi) \\ &\approx \int_1^\infty a s^{2-\tau} \mathcal{G}_\pm(s/s_\xi)\, ds\end{aligned} \tag{1.6.2}$$

Substituting $u = s/s_\xi$, that is $s = s_\xi u$ and $ds = s_\xi du$. With the new lower integration limit $1/s_\xi$ we have

$$\begin{aligned}p_c\chi(p) &= \int_{1/s_\xi}^\infty a\,(s_\xi u)^{2-\tau}\, \mathcal{G}_\pm(u) s_\xi\, du \\ &= s_\xi^{3-\tau} a \int_{1/s_\xi}^\infty u^{2-\tau} \mathcal{G}_\pm(u)\, du \\ &= |p - p_c|^{-(3-\tau)/\sigma} a b^{3-\tau} \int_0^\infty u^{2-\tau} \mathcal{G}_\pm(u)\, du \quad \text{for } p \to p_c\end{aligned}$$

where we, in the last step, have substituted $s_\xi = b|p - p_c|^{-1/\sigma}$ for $p \to p_c$ and changed the lower limit to zero as s_ξ diverges at $p = p_c$.

(i) Assume $p < p_c$. Then, in the limit $p \to p_c^-$,

$$\chi(p) = (p_c - p)^{-(3-\tau)/\sigma} \frac{a b^{3-\tau}}{p_c} \int_0^\infty u^{2-\tau} \mathcal{G}_-(u)\, du \tag{1.6.3}$$

with

$$\gamma^- = \frac{3-\tau}{\sigma} \tag{1.6.4a}$$

$$\Gamma^- = \frac{ab^{3-\tau}}{p_c} \int_0^\infty u^{2-\tau} \mathcal{G}_-(u)\, du. \tag{1.6.4b}$$

(ii) Assume $p > p_c$. Then, in the limit $p \to p_c^+$,

$$\chi(p) = (p - p_c)^{-(3-\tau)/\sigma} \frac{ab^{3-\tau}}{p_c} \int_0^\infty u^{2-\tau} \mathcal{G}_+(u)\, du \tag{1.6.5}$$

with

$$\gamma^+ = \frac{3-\tau}{\sigma} \tag{1.6.6a}$$

$$\Gamma^+ = \frac{ab^{3-\tau}}{p_c} \int_0^\infty u^{2-\tau} \mathcal{G}_+(u)\, du. \tag{1.6.6b}$$

(iii) (a) By inspection $\gamma^- = \gamma^+ = (3-\tau)/\sigma$.
(b) The ratio of critical amplitudes

$$\frac{\Gamma^+}{\Gamma^-} = \frac{\int_0^\infty u^{2-\tau} \mathcal{G}_-(u)\, du}{\int_0^\infty u^{2-\tau} \mathcal{G}_+(u)\, du} \tag{1.6.7}$$

is independent of the proportionality constants a and b and p_c and only depends on the universal critical exponent τ and the universal scaling functions $\mathcal{G}_\pm$. Thus the ration Γ^+/Γ^- is itself universal.
(c) The ratio of the critical amplitudes Γ^+/Γ^- is related to the distance between the numerical results for the average cluster size for $p < p_c$ and $p > p_c$ respectively. Numerical simulations confirm that Γ^+/Γ^- is universal and one finds $\Gamma^+/\Gamma^- \approx 200$ using the numerical results displayed.