1.8 *Finite-size scaling and scaling function for the average cluster size.*

(i) The average cluster size is by definition

$$\chi(p; L = \infty) = \frac{\sum_{s=1}^{\infty} s^2 n(s,p)}{\sum_{s=1}^{\infty} s n(s,p)}. \qquad (1.8.1)$$

For $p \to p_c$, the denominator approaches the constant p_c. Substituting the sum with an integral we find

$$\begin{aligned}\chi(p; L = \infty) &\propto \int_1^{\infty} s^2 n(s,p)\, ds \\ &= \int_1^{\infty} s^{2-\tau} \mathcal{G}\left(s/s_\xi\right)\, ds \\ &= \int_{1/s_\xi}^{\infty} (u s_\xi)^{2-\tau} \mathcal{G}(u) s_\xi \, du \\ &= s_\xi^{3-\tau} \int_{1/s_\xi}^{\infty} u^{2-\tau} \mathcal{G}(u)\, du \\ &\propto |p - p_c|^{-(3-\tau)/\sigma} \qquad (1.8.2)\end{aligned}$$

as for $p \to p_c$, $s_\xi \to \infty$ and the integral approaches a constant number. Thus

$$\gamma = \frac{3-\tau}{\sigma}. \qquad (1.8.3)$$

(ii) For $p \to p_c$, the correlation length

$$\xi(p) \propto |p - p_c|^{-\nu} \Rightarrow |p - p_c| \propto \xi^{-\frac{1}{\nu}}, \qquad (1.8.4)$$

that is,

$$\chi(\xi; L = \infty) \propto |p - p_c|^{-\gamma} \propto \xi^{\gamma/\nu} \quad \text{for } p \to p_c. \qquad (1.8.5)$$

(iii) (a) There are only two relevant length scales in the problem, the correlation length ξ and the lattice size L. When $L \ll \xi$, L will be the limiting length scale taking the role of ξ and thus

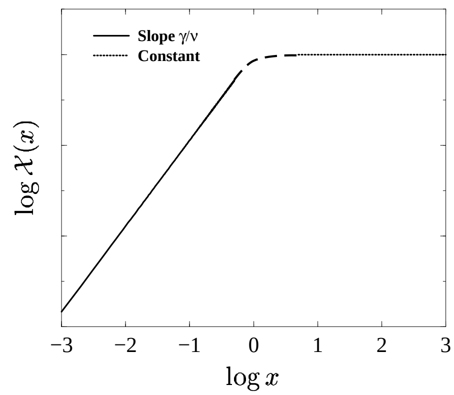
$$\chi(\xi; L) \propto L^{\gamma/\nu} \quad \text{for } p \to p_c, 1 \ll L \ll \xi.. \qquad (1.8.6)$$

(b) When $L \gg \xi$, we 'do not know' that the lattice is finite and $\chi(\xi; L)$ will not be affected by the finite size of the lattice. Thus we have

$$\chi(\xi;L) = \begin{cases} \xi^{\gamma/\nu} & \text{for } L \gg \xi \\ L^{\gamma/\nu} & \text{for } L \ll \xi \end{cases} = \xi^{\gamma/\nu}\mathcal{X}(L/\xi) \tag{1.8.7}$$

where the scaling function

$$\mathcal{X}(x) = \begin{cases} constant & \text{for } x \gg 1 \\ x^{\gamma/\nu} & \text{for } x \ll 1. \end{cases} \tag{1.8.8}$$



(c) If $p = p_c$, the correlation length $\xi = \infty$ and we are always in the case $L \ll \xi$ ($x \ll 1$). Thus by plotting $\log \chi(L,\infty)$ versus $\log L$ we get a straight line with slope γ/ν.

(iv) At $p = p_c, \xi = \infty$ so according to Equation (1.8.6) the average cluster size $\chi(\xi = \infty; L) \propto L^{\gamma/\nu}$. Hence we find

$$\begin{aligned}\chi(\xi=\infty;L) &\propto \int_1^\infty s^2 n(s,p_c,L)\,ds \\ &= \int_1^\infty s^{2-\tau}\tilde{\mathcal{G}}\left(s/L^D\right)\,ds \\ &= \int_{1/L^D}^\infty (uL^D)^{2-\tau}\tilde{\mathcal{G}}(u)L^D\,du \\ &= L^{D(3-\tau)}\int_{1/L^D}^\infty u^{2-\tau}\tilde{\mathcal{G}}(u)\,du \\ &\propto L^{D(3-\tau)} \end{aligned} \tag{1.8.9}$$

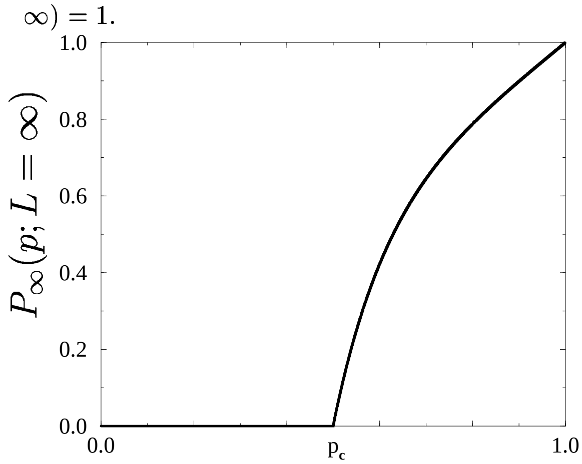
as for $L \gg 1$ the integral approaches a number. Thus

$$\frac{\gamma}{\nu} = D(3-\tau). \tag{1.8.10}$$

1.9 *Finite-size scaling and scaling function for the order parameter.*

(i) (a) The order parameter $P_\infty(p; L = \infty)$ is the probability that (at occupation probability p) an arbitrary site belongs to the percolating infinite cluster.

(b) For $p \le p_c$ there are no percolating infinite clusters, so $P_\infty(p; L = \infty) = 0$. The critical occupation probability is the concentration p_c above which a percolating infinite cluster appears for the first time and the order parameter becomes nonzero for $p > p_c$. Clearly $P_\infty(p = 1; L = \infty) = 1$.



(c) The probability that an arbitrary site belongs to an s-cluster is $sn(s, p; L = \infty)$. The probability that an arbitrary site belongs to any finite cluster is $\sum_{s=1}^{\infty} sn(s, p; L = \infty)$. The relation thus states that for a given site

$$P(\text{in infinite cluster}) = P(\text{occupied}) \;-\; P(\text{in finite cluster})$$

$$P_\infty(p; L = \infty) = p - \sum_{s=1}^{\infty} sn(s, p; L = \infty).$$

(ii) (a) For $p \to p_c^+$, the correlation length

$$\xi(p) \propto (p - p_c)^{-\nu} \Rightarrow (p - p_c) \propto \xi^{-\frac{1}{\nu}}, \qquad (1.9.1)$$

that is,

$$P_\infty(\xi; L = \infty) \propto (p - p_c)^{\beta} \propto \xi^{-\beta/\nu}. \qquad (1.9.2)$$

(b) There are only two relevant length scales in the problem, the correlation length ξ and the lattice size L. When $1 \ll L \ll \xi$, L will be the limiting length scale taking the role of ξ and thus

$$P_\infty(\xi; L) \propto L^{-\beta/\nu} \quad \text{for } 1 \ll L \ll \xi, p \to p_c. \qquad (1.9.3)$$

(c) If $p = p_c$, the correlation length $\xi = \infty$ and we are always in the case $L \ll \xi$ ($x \ll 1$). Thus by plotting $\log P_\infty(\infty; L)$ versus $\log L$ we get a straight line with slope $-\beta/\nu$.

(iii) (a) The order parameter at $p = p_c$ in an infinite lattice is zero. In the limit $L \to \infty$, the cluster number density $n(s, p_c; L)$ tends to $s^{-\tau} g(0)$, so also the right hand side equals zero.

(b) At $p = p_c, \xi = \infty$ so according to Equation (1.9.3) the order parameter $P_\infty(\xi = \infty, L) \propto L^{-\beta/\nu}$. Using Equation (1.95) in the question and substituting the scaling law for the cluster number density we find

$$\begin{aligned}
P_\infty(\xi = \infty; L) &= \sum_{s=1}^{\infty} s^{1-\tau}[g(0) - g(s/L^D)] \\
&\propto \int_1^\infty s^{1-\tau}[g(0) - g(s/L^D)]\, ds \qquad \text{main contribution from } s \gg 1 \\
&= \int_{1/L^D}^\infty (uL^D)^{1-\tau}[g(0) - g(u)]L^D\, du \qquad u = s/L^D \\
&= L^{D(2-\tau)} \int_{1/L^D}^\infty u^{1-\tau}[g(0) - g(u)]\, du \\
&\propto L^{D(2-\tau)} \qquad (1.9.4)
\end{aligned}$$

as for $L \gg 1$ the integral approaches a number (lower limit approaches zero). Thus

$$P_\infty(\xi = \infty; L) \propto L^{-\beta/\nu} \propto L^{D(2-\tau)} \quad \text{for } L \gg 1, \qquad (1.9.5)$$

implying the scaling relation

$$-\frac{\beta}{\nu} = D(2 - \tau). \qquad (1.9.6)$$

1.13 *Renormalisation and finite-size scaling of the cluster no. density.*

(i) The square of the radius of gyration $R^2(s)$ for a given s-cluster is defined as the average square distance to the centre of mass,

$$R^2(s) = \frac{1}{s}\sum_{i=1}^{s} |r_i - r_{cm}|^2,$$

where r_i denotes the position of the ith-particle and r_{cm} the centre of mass. The radius of gyration R_s^2 is the average of $R^2(s)$ over all s-clusters, that is,

$$R_s = \sqrt{\langle R^2(s)\rangle}. \tag{1.13.1}$$

The radius of gyration R_s measures the linear extent of an s-cluster. Thus if $\ell \gg R_s$, the finite cluster is contained within the box of size ℓ implying $M(\ell, R_s) = s$. If $\ell \ll R_s$, it appears as if the cluster in infinite (we don't know it is finite). At $p = p_c$, the cluster is fractal with $M(\ell, R_s) \propto \ell^D$, D being the fractal dimension of the infinite percolating cluster. Thus

$$M(\ell, R_s) \propto \begin{cases} \ell^D & \text{for } \ell \ll R_s, \\ s \propto R_s^D & \text{for } \ell \gg R_s \end{cases} \tag{1.13.2}$$

since the mass of the infinite cluster at $p = p_c$ is proportional to ℓ^D, it is natural to assume that also $s \propto R_s^D$. Thus

$$M(\ell, R_s) = \ell^D m(\ell/R_s), \tag{1.13.3}$$

whit a crossover function

$$m(x) \propto \begin{cases} \text{constant} & \text{for } x \ll 1 \\ x^{-D} & \text{for } x \gg 1 \end{cases} \tag{1.13.4}$$

that is, $D_1 = D$ and $D_2 = 1$.

(ii) From above, we have $M(\ell, R_s) = R_s^D$ for $\ell \gg R_s$. The real space renormalisation transformation renormalises all length scales by a factor b, e.g., $R_s \to R_s/b$. Thus

$$s' = M(\ell/b, R_s/b) = (R_s/b)^D = R_s^D/b^D = s/b^D \tag{1.13.5}$$

where we have used $\ell \gg R_s \Rightarrow \ell/b \gg R_s/b$.

(iii) $sn(s, p_c; L)$ is the probability that a site belongs to a cluster of size s in a lattice of linear size L *per lattice site* while $s'n(s', p_c; L/b)$ is the probability that a site belongs to a cluster of size s' in a lattice of linear size L/b *per lattice site*.
As the number of clusters in the original and renormalised lattice is the same we have

$$\begin{aligned} L^d sn(s, p_c; L) &= (L/b)^d s'n(s', p_c; L/b) \Rightarrow \\ sn(s, p_c; L) &= b^{-d} s'n(s', p_c; L/b), \end{aligned} \tag{1.13.6}$$

with $s' = s/b^D$, see question (ii).

(iv) Given the scaling form of the cluster number density

$$n(s,p) = s^{-\tau}\mathcal{G}(s/s_\xi) \quad \text{for } p \to p_c, s \gg 1. \tag{1.13.7}$$

As the characteristic cluster size $s_\xi \propto \xi^D$ where the correlation length $\xi \propto |p - p_c|^{-\nu}$ we find

$$n(s,p) = s^{-\tau}\mathcal{G}(s/\xi^D). \tag{1.13.8}$$

In a finite system at $p = p_c$ where $L \ll \xi = \infty$, one would thus, using a finite-size scaling argument, expect

$$n(s,p) = s^{-\tau}\tilde{\mathcal{G}}(s/L^D). \tag{1.13.9}$$

For $s/L^D \ll 1 \Leftrightarrow s \ll L^D$ (i.e. $L \to \infty$), the cluster number density must be independent of system size, leaving

$$n(s, p_c) \propto s^{-\tau} \Rightarrow \tilde{\mathcal{G}}(x) = \text{constant} \quad x \ll 1. \tag{1.13.10}$$

Clearly $s/L^D \gg 1 \Leftrightarrow s \gg L^D$ is very unlikely, so $\tilde{\mathcal{G}}(x)$ decays rapidly for $x \gg 1$.

(v) Combining Equations (1.13.8) (1.13.6) and (1.13.5) we find

$$\begin{aligned} s^{1-\tau}\mathcal{G}(s/L^D) &= b^{-d}(s/b^D)(s/b^D)^{-\tau}\mathcal{G}\left(\frac{s/b^D}{(L/b)^D}\right) \\ &= b^{-d-D+D\tau}s^{1-\tau}\mathcal{G}(s/L^D), \end{aligned} \tag{1.13.11}$$

from which we conclude

$$-d - D + D\tau = 0, \tag{1.13.12}$$

implying the scaling relation

$$\tau = \frac{d+D}{D}. \tag{1.13.13}$$