1.11 *Real-space RG transformation on a square lattice.*

(i) The real space renormalisation technique is based on a so-called *block site technique* and has three basic steps:

1. *Divide the lattice into blocks of linear size b.*
2. *Next, the coarse graining procedure takes place. The sites in the blocks are* average in some way *and the* **entire** *bloch is replaced by a single (super) site which is occupied with a probability according to the renormalisation group transformation R_b.*
Important to keep the symmetry of the original lattice such that the coarse graining procedure can be repeated. The two operations creates a new lattice with lattice constant b times as large as in the original lattice.
3. *Restore original lattice constant by rescaling length scales by the factor b.*

The coarse graining procedure in step 2 is to eliminate from the system all fluctuations whose scale is smaller than the block size b, so we eventually will explore the large scale behaviour of the system.

(ii) It is always a good idea to draw the situation.

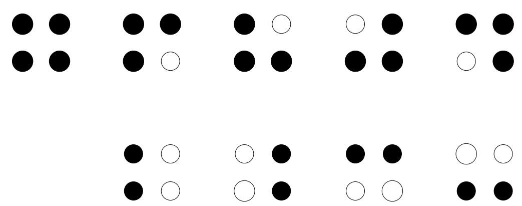


Fig. 1.11.1 The block of size 2×2 contains a spanning cluster if all 4 sites are occupied – probability p^4 – or three sites are occupied, one empty – probability $4p^3(1-p)$ – as there a four different ways of placing the empty site. Also, four different configucations contain a spanning cluster if 2 sites are occupied and 2 sites empty – probability $4p^2(1-p)^2$.

Thus

$$R_b(p) = p^4 + 4p^3(1-p) + 4p^2(1-p)^2 = p^4 - 4p^3 + 4p^2. \quad (1.11.1)$$

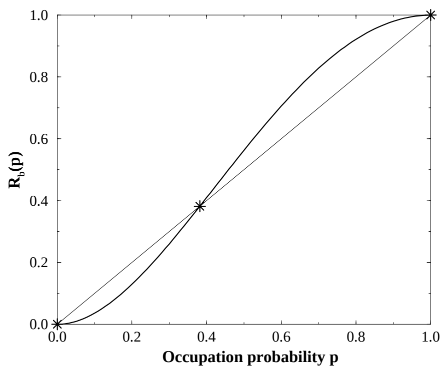


Fig. 1.11.2 The crosssection between the graph of the renormalisation group transformation $R_b(p)$ and the identity transformation $R_b(p) = p$ are solutions to the fixed point equation.

(iii) Solving the equation graphically yields

$$p^\star = \begin{cases} 0 & \text{trivial fixed point} \\ 1 & \text{trivial fixed point} \\ 0.382 & \text{non-trivial fixed point.} \end{cases}$$

The correlation lengths $\xi = 0$ for the trivial fixed points $p^\star = 0$ and $p^\star = 1$ corresponding to the empty and fully occupied lattice, respectively.
The non-trivial fixed point $p^\star = 0.382$ is accociated with the critical occupation probability p_c where the correlation length is infinite.
When performing the real space renormalisation procedure, length scales are rescaled by the factor b. Thus, the correlation length $\xi \to \xi/b$ only remains invariant if $\xi = 0$, associated with the trivial fixed points or $\xi = \infty$, associated with the non-trivial fixed point. As $\xi \propto |p - p_c|^{-\nu}$, the non-trivial fixed point is identified as p_c.

(iv) We identify $p_c = p^\star = 0.382$ (see (iii)). Let A be a constant. Then

$$\xi = A\,|p - p_c|^{-\nu} \tag{1.11.2a}$$
$$\xi' = A\,|R_b(p) - p_c|^{-\nu}. \tag{1.11.2b}$$

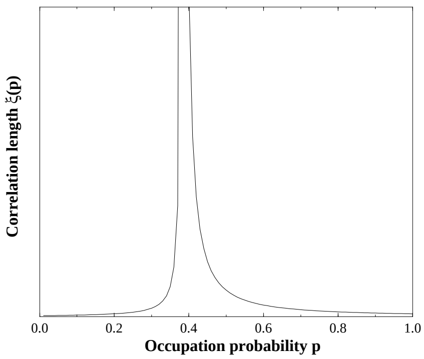


Fig. 1.11.3 If we start out with a finite correlation length, the rescaled correlation length $\xi' = \xi/b$ will decrease ($b > 1$) with an associated flow in p-space. Starting out with $0 < p < p_c$, the flow will be toward the trivial fixed point $p = 0$ where $\xi = 0$. If we start out with $1 > p > p_c$, the flow will be toward the trivial fixed point $p = 1$ where $\xi = 0$. If we start at the nontrivial fixed point $p^\star$, we will remain as $\xi = \infty$.

As $\xi' = \xi/b$ we find

$$\begin{aligned}|p - p_c|^{-\nu} &= b|R_b(p) - p_c|^{-\nu} \\ &= b|R_b(p) - R(p_c)|^{-\nu},\end{aligned} \tag{1.11.3}$$

from which we find for $p \to p_c$

$$\nu = \frac{\log b}{\log\left(\frac{dR_b(p_c)}{dp}\right)}. \tag{1.11.4}$$

Now

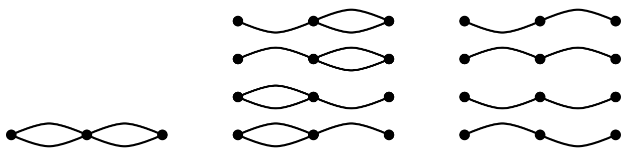
$$\begin{aligned}\frac{dR_b(p_c)}{dp} &= (4p^3 - 12p^2 + 8p)|_{p^\star=0.382} \\ &= 1.53 \Rightarrow \\ \nu &= \frac{\log 2}{\log 1.53} = 1.63.\end{aligned} \tag{1.11.5}$$

The exact values in $d = 2$ are $p_c = 0.592746\ldots$ and $\nu = 4/3$, respectively. The discrepancy comes about as the real space renormalisation procedure considered above is approximative. E.g. might the real space renormalisation procedure split a cluster into two or more clusters. In general, one would have to introduce more parameters for each renormalisation step to avoid the approximative nature of the above process which is truncated to consider only one parameter, the occupation probability p.

(v) Quantities which are universal are independent of the microscopic details such as the underlying lattice structure and depend only one the dimensionality of the problem at hand. Examples are the critical exponents, such as τ, ν, and σ, describing the behaviour of quantities close to the phase transition and scaling functions. The universality comes about, as it is the large scale behaviour which determine these quantities. Non-universal quantities will depend on the lattice structure as e.g. the critical occupation probability p_c.

1.12 *Real-space renormalisation group transformation on a square lattice.*

(i) There are nine configurations that have a connected path from **A** to **B**:



p^4 $\quad 4p^3(1-p) \quad 4p^2(1-p)^2$

Adding the probabilities for these configurations, we find

$$
\begin{aligned}
R_b(p) &= p^4 + 4p^3(1-p) + 4p^2(1-p)^2 \\
&= p^4 - 4p^3 + 4p^2. \qquad (1.12.1)
\end{aligned}
$$

(ii) (a) The fixed point Equation $R_b(p) = p$ is solved graphically by plotting the graph of $R_b(p)$ versus p and locating the intersections with the line $R_b(p) = p$.
By inspection, we find the three fixed points

$$
p^\star = \begin{cases} 0 & \text{trivial fixed point - empty lattice} \\ 0.38 & \text{non-trivial fixed point} \\ 1 & \text{trivial fixed point - fully occupied lattice.} \end{cases}
$$

(b) When performing the real-space renormalisation procedure, length scales are rescaled by the factor b.
If we start out with a finite correlation length, the rescaled correlation length $\xi' = \xi/b$ will decrease $(b > 1)$ with an

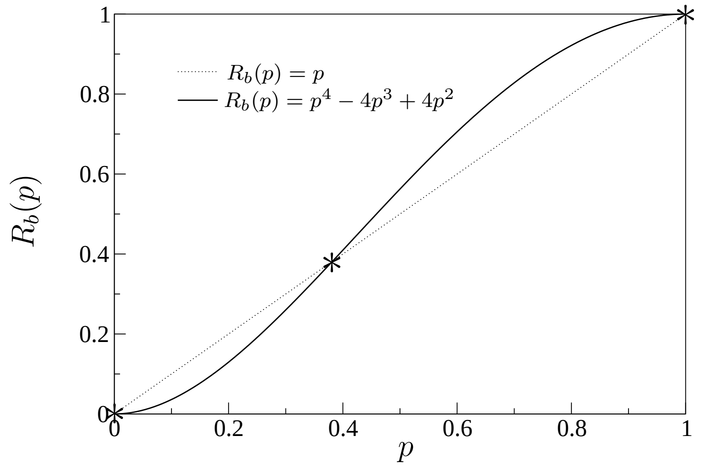


Fig. 1.12.1 The fixed point Equation $R_b(p^\star) = p^\star$ are $p^\star = 0, 0.38, 1$.

associated flow in p-space as indicated below. Starting out with $p < p^\star$, the flow will be toward $p^\star = 0$. If we started out with $p > p^\star$, the flow will be toward $p^\star = 1$.

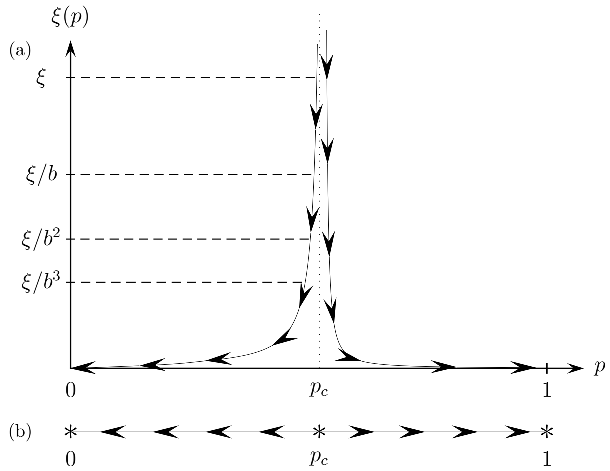


Fig. 1.12.2 (a) A sketch of the correlation length as a function of occupation probability. The dotted line shows the position of p_c. (b) The corresponding flow in parameter space.

(c) The correlation length $\xi \rightarrow \xi/b$ only remains invariant if $\xi = 0$, associated with the trivial fixed points $p^\star = 0$ (empty lattice) or $p^\star = 1$ (fully occupied lattice) or $\xi = \infty$, associated with the non-trivial fixed point $p^\star \approx 0.38$. Since the correlation length is $\xi = 0$ or $\xi = \infty$ at the fixed point, there is no characteristic scale and scale invariance prevails.

(iii) (a) Let A denote a constant. Then

$$\xi = A\,|p - p_c|^{-\nu} \tag{1.12.2a}$$

$$\xi' = A\,|R_b(p) - p_c|^{-\nu}. \tag{1.12.2b}$$

As $\xi' = \xi/b$ we find

$$|p - p_c|^{-\nu} = b|R_b(p) - p_c|^{-\nu} = b|R_b(p) - R(p_c)|^{-\nu},$$

from which we find for $p \to p_c$

$$\nu = \frac{\log b}{\log\left(\frac{dR_b(p_c)}{dp}\right)}.$$

(b) Now

$$\frac{dR_b}{dp}|_{p^\star} = (4p^3 - 12p^2 + 8p)|_{p^\star=0.38}$$
$$\approx 1.53 \tag{1.12.3}$$

and hence

$$\nu = \frac{\log 2}{\log 1.53} \approx 1.63. \tag{1.12.4}$$

The exact values in $d = 2$ are $p_c = 0.5$ and $\nu = 4/3$,

Exercises

2.1 *The entropy and the free energy.*

(i) According to the Boltzmann's distribution, the probability p_r to find an equilibrium system in a microstate r with energy E_r at temperature T is given by

$$p_r = \frac{\exp(-\beta E_r)}{\sum_r \exp(-\beta E_r)} = \frac{1}{Z} \exp(-\beta E_r) \tag{2.1.1}$$

where $\beta = 1/(k_B T)$ and Z denotes the partition function. Therefore, the entropy

$$\begin{aligned} S &= -k_B \sum_r p_r \ln p_r \\ &= -k_B \sum_r \frac{1}{Z} \exp(-\beta E_r) \left[\ln\left(\exp(-\beta E_r)\right) - \ln Z\right] \\ &= k_B \ln Z - k_B \sum_r \frac{(-\beta E_r)\exp(-\beta E_r)}{Z} \\ &= k_B \ln Z + \frac{1}{T} \sum_r \frac{E_r \exp(-\beta E_r)}{Z} \\ &= k_B \ln Z + \frac{\langle E \rangle}{T}. \end{aligned}$$

(ii) From part (i) we find

$$\ln Z = \frac{1}{k_B}\left(S - \frac{\langle E \rangle}{T}\right), \tag{2.1.2}$$

so the free energy

$$F = -k_B T \ln Z = -T\left(S - \frac{\langle E \rangle}{T}\right) = \langle E \rangle - TS. \tag{2.1.3}$$

2.2 *Fluctuation-dissipation theorem.*
First we note that the average total energy

$$\langle E \rangle = -\left(\frac{\partial \ln Z}{\partial \beta}\right)_H, \tag{2.2.1}$$

since

$$
\begin{aligned}
-\left(\frac{\partial \ln Z}{\partial \beta}\right)_H &= -\frac{1}{Z}\frac{\partial Z}{\partial \beta} \\
&= -\frac{1}{Z}\frac{\partial}{\partial \beta}\left(\sum_{\{s_i\}} \exp(-\beta E_{\{s_i\}})\right) \\
&= \frac{1}{Z}\sum_{\{s_i\}} \exp(-\beta E_{\{s_i\}}) E_{\{s_i\}} \qquad (2.2.2)
\end{aligned}
$$

However, the *instantaneous* total energy will, of course, fluctuate around the average total energy. The magnitude of the fluctuations is determined by the standard deviation ΔE where

$$(\Delta E)^2 = \langle (E - \langle E\rangle)^2\rangle = \langle E^2 + \langle E\rangle^2 - 2E\langle E\rangle\rangle = \langle E^2\rangle - \langle E\rangle^2.$$

Differentiating twice $\ln Z$ with respect to β we find

$$
\begin{aligned}
\left(\frac{\partial^2 \ln Z}{\partial \beta^2}\right)_H &= -\frac{\partial}{\partial \beta}\left(-\frac{\partial \ln Z}{\partial \beta}\right)_H \\
&= -\frac{\partial}{\partial \beta}\left(\frac{1}{Z}\sum_{\{s_i\}} \exp(-\beta E_{\{s_i\}}) E_{\{s_i\}}\right)_H \\
&= \frac{1}{Z}\sum_{\{s_i\}} \exp(-\beta E_{\{s_i\}}) E^2_{\{s_i\}} + \frac{1}{Z^2}\left(\frac{\partial Z}{\partial \beta}\right)_H \sum_{\{s_i\}} \exp(-\beta E_{\{s_i\}}) E_{\{s_i\}} \\
&= \frac{1}{Z}\sum_{\{s_i\}} \exp(-\beta E_{\{s_i\}}) E^2_{\{s_i\}} + \left(\frac{\partial \ln Z}{\partial \beta}\right)_H \frac{1}{Z}\sum_{\{s_i\}} \exp(-\beta E_{\{s_i\}}) E_{\{s_i\}} \\
&= \langle E^2\rangle - \langle E\rangle^2. \qquad (2.2.3)
\end{aligned}
$$

However,

$$\left(\frac{\partial^2 \ln Z}{\partial \beta^2}\right)_H = -\left(\frac{\partial \langle E\rangle}{\partial \beta}\right)_H = -\left(\frac{\partial \langle E\rangle}{\partial T}\right)_H \frac{\partial T}{\partial \beta} = -C\frac{\partial (k_B\beta)^{-1}}{\partial \beta} = k_B T^2 C,$$

where C denotes the heat capacity at constant external parameter, such that

$$k_B T^2 C = \langle E^2\rangle - \langle E\rangle^2. \qquad (2.2.4)$$