SECTION B

3.

(i) (a) Kadanoff argued that since spins are correlated over scales up to the correlation length ξ, it may be plausible to regard spins within regions up to ξ in size as behaving like a single block spin. In this spirit, Kadanoff outlined a real-space renormalisation procedure over scales $b \lesssim \xi$.

1. Divide the lattice into blocks of linear size b (in terms of the lattice constant) with each block I containing b^2 spins.
2. Replace each block I of spins by a single block spin s_I according to some coarse graining rule which is some function of the spins within block I.
3. Rescale all lengths by the factor b to restore the original lattice spacing.

(b) Let $E_{\{s_i\}}$ denote the energy of the Ising model and s_I the Kadanoff block spins in the coarse grained system and t and h the reduced temperature and external field, respectively. Then the partition function

$$\begin{aligned} Z(N,t,h) &= \sum_{\{s_i\}} \exp\left(-\beta E_{\{s_i\}}\right) \\ &= \sum_{\{s_I\}} \sum_{\substack{\text{configurations } \{s_i\} \\ \text{consistent with } \{s_I\}}} \exp\left(-\beta E_{\{s_i\}}\right) \end{aligned}$$

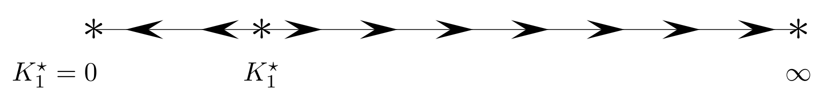
Defining the energy of the Ising model in the renormalised lattice, $E'_{\{s_I\}}$, through the equation

$$\exp\left(-\beta E'_{\{s_I\}}\right) = \sum_{\substack{\text{configurations } \{s_i\} \\ \text{consistent with } \{s_I\}}} \exp\left(-\beta E_{\{s_i\}}\right),$$

we have that the partition function remains invariant under the real-space renormalisation transformation

$$\begin{aligned} Z(N,t,h) &= \sum_{\{s_I\}} \sum_{\substack{\text{configurations } \{s_i\} \\ \text{consistent with } \{s_I\}}} \exp\left(-\beta E_{\{s_i\}}\right) \\ &= \sum_{\{s_I\}} \exp\left(-\beta E'_{\{s_I\}}\right) \\ &= Z(N',t',h'). \end{aligned}$$

(ii) (a) The system in the first column is flowing towards the low-temperature fixed point $K_1^\star = \infty$ in which all spins are aligned, implying $T_1 < T_c$. In the second column, the systems are self-similar and $T_2 = T_c$. The system in the third column is flowing towards the high-temperature fixed point $K_1^\star = 0$ where spins are pointing up and down randomly implying $T_3 > T_c$. In the Figure below we sketch the associated flow in the reduced nearest-neighbour coupling constant $K_1 = J/k_B T$. The trivial fixed points $K_1^\star = \infty$ and $K_1^\star = 0$ are stable while the non-trivial fixed point $K_1^\star$ is unstable.



(b) In the low-temperature fixed point, $K_1^\star = \infty$, all spins are aligned. The two possible microstates are trivially self-similar and the correlation length $\xi = 0$. In the high-temperature fixed point, $K_1^\star = \infty$, spins are non-interacting, pointing up and down randomly. The possible microstates are trivially self-similar and the correlation length $\xi = 0$. The non-trivial fixed point consists of self-similar microstates where spins are correlated over all scales with a correlation length $\xi = \infty$.

(c) The correlation length is the only relevant length scale for the Ising model (apart from the lattice spacing). Since the fixed points of the renormalisation translation are related to self-similar microstates, there is no relevant length scale associated with the fixed point. The correlation length is either $\xi = 0$ at the two trivial, stable fixed points or $\xi = \infty$ at the nontrivial, unstable fixed point. Hence scale invariance prevails.

(iii) (a) Upon renormalisation, the correlation length is reduced by a factor b,

$$\xi' = \frac{\xi}{b}.$$

In zero external field, the correlation length $\xi = \text{const.}|t|^{-\nu}$ as $t \to 0^\pm$, so that

$$\text{const.}|t'|^{-\nu} = \frac{\text{const.}|t|^{-\nu}}{b} \quad \text{for } t \to 0^\pm.$$

Substituting $t' = b^{y_t} t$ and rearranging, we find

$$b^{1-\nu y_t} = 1.$$

Since b is arbitrary, the exponent must be zero and we conclude that

$$y_t = \frac{1}{\nu}.$$

(b) Assuming that

$$f(t,h) = b^{-2} f(b^{y_t} t, b^{y_h} h) \quad \text{for } t \to 0^\pm, h \to 0.$$

Let $b = |t|^{-1/y_t} = |t|^{-\nu} \propto \xi$. Then

$$\begin{aligned} f(t,h) &= \left[|t|^{-1/y_t}\right]^{-2} f\left(t\left[|t|^{-1/y_t}\right]^{y_t}, h\left[|t|^{-1/y_t}\right]^{y_h}\right) \\ &= |t|^{2/y_t} f\left(t/|t|, h/|t|^{y_h/y_t}\right) \\ &= |t|^{2/y_t} f\left(\pm 1, h/|t|^{y_h/y_t}\right). \end{aligned}$$

Making the identifications

$$2 - \alpha = 2/y_t \tag{3.1a}$$

$$\Delta = y_h/y_t \tag{3.1b}$$

$$\mathcal{F}_\pm(h/|t|^\Delta) = f\left(\pm 1, h/|t|^{y_h/y_t}\right), \tag{3.1c}$$

we recover the Widom scaling ansatz

$$f_s(t,h) = |t|^{2-\alpha} \mathcal{F}_\pm(h/|t|^\Delta) \quad \text{for } t \to 0^\pm, h \to 0.$$

The two branches of the scaling function for the free energy for $t \to 0^\pm$ appear naturally as a result of the first argument in the free energy on the right-hand side of Equation (3.1c).