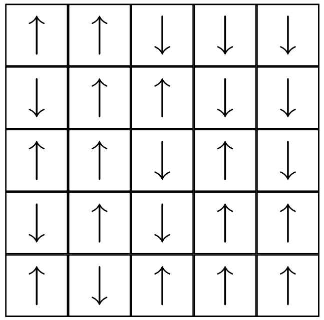
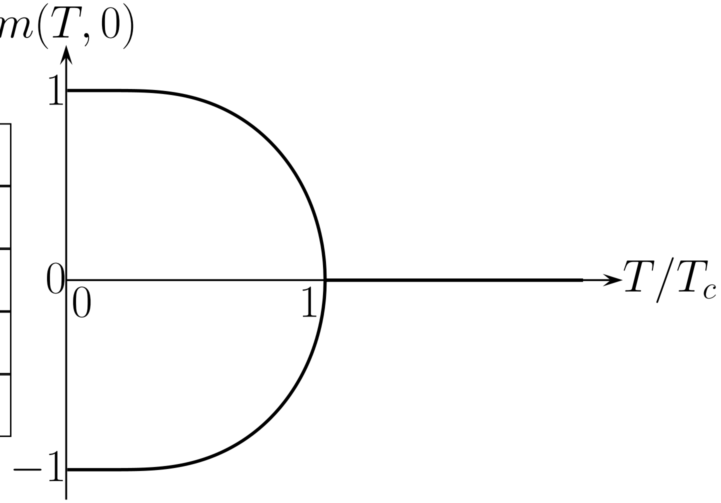
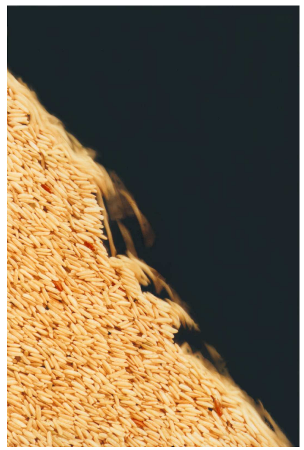
# Complexity and Criticality

- **Criticality**
  - Ising model
  - Phase transition at $(T_c, 0)$
  - Scale invariance and £xed points
- **Complexity**
  - Earthquakes and rainfall
  - Rice-pile experiment
  - Oslo rice-pile model
  - Self-organised criticality





The simplest model of a ferromagnet consists of $N$ spins $s_i = \pm 1 = \uparrow$ or $\downarrow$, $i = 1, \ldots, N$ with constant nearest-neighbour interaction $J > 0$ placed in a uniform external £eld $H$. The energy of microstate $\{s_i\} = \{s_1, s_2, \ldots, s_N\}$

$$E_{\{s_i\}} = \text{spin-spin interaction} + \text{spin-external £eld interaction}$$

$$= -J \sum_{\langle ij \rangle} s_i s_j - H \sum_{i=1}^{N} s_i .$$

| | | | | |
|---|---|---|---|---|
| $\uparrow$ | $\uparrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ |
| $\downarrow$ | $\uparrow$ | $\uparrow$ | $\downarrow$ | $\downarrow$ |
| $\uparrow$ | $\uparrow$ | $\downarrow$ | $\uparrow$ | $\downarrow$ |
| $\downarrow$ | $\uparrow$ | $\downarrow$ | $\uparrow$ | $\uparrow$ |
| $\uparrow$ | $\downarrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ |

The partition function

$$Z(T, H) = \sum_{\{s_i\}} \exp\left(-\beta E_{\{s_i\}}\right) .$$

The free energy per spin

$$f(T, H) = -\frac{1}{N} k_B T \ln Z .$$

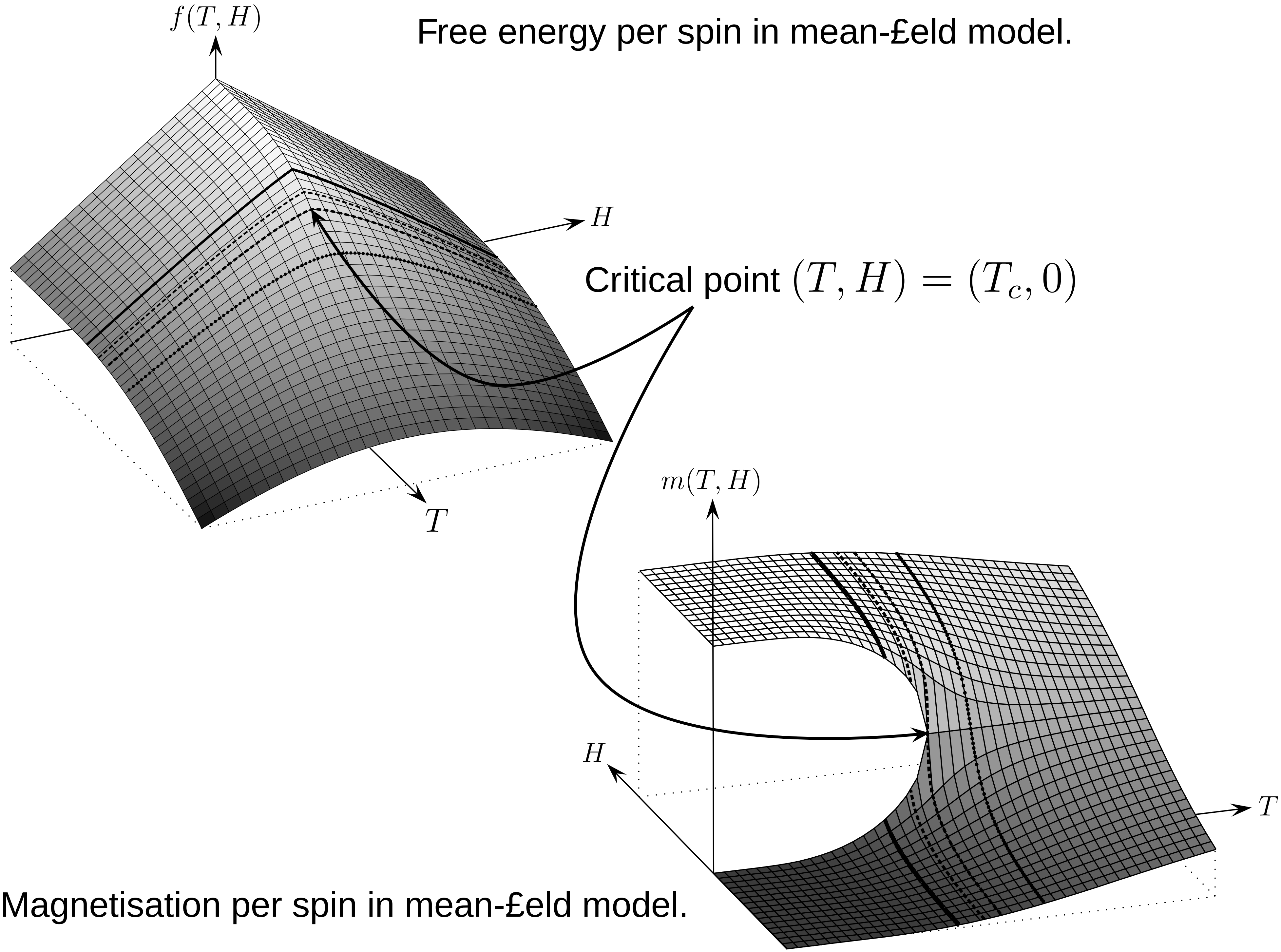
The magnetisation per spin

$$m(T, H) = -\left(\frac{\partial f}{\partial H}\right)_T .$$

The susceptibility per spin

$$\chi(T, H) = \left(\frac{\partial m}{\partial H}\right)_T .$$

Free energy per spin in mean-£eld model.




Critical point $(T, H) = (T_c, 0)$

Magnetisation per spin in mean-£eld model.

## Phase transition at $(T, H) = (T_c, 0)$



Assume $H = 0$. In equilibrium, the free energy is minimised

$$F = \langle E \rangle - TS.$$

$T = 0$: Energy minimised: spins are aligned: $m(0, 0) = \pm 1$.

Con£gurations are self-similar. The correlation length $\xi(0, 0) = 0$.

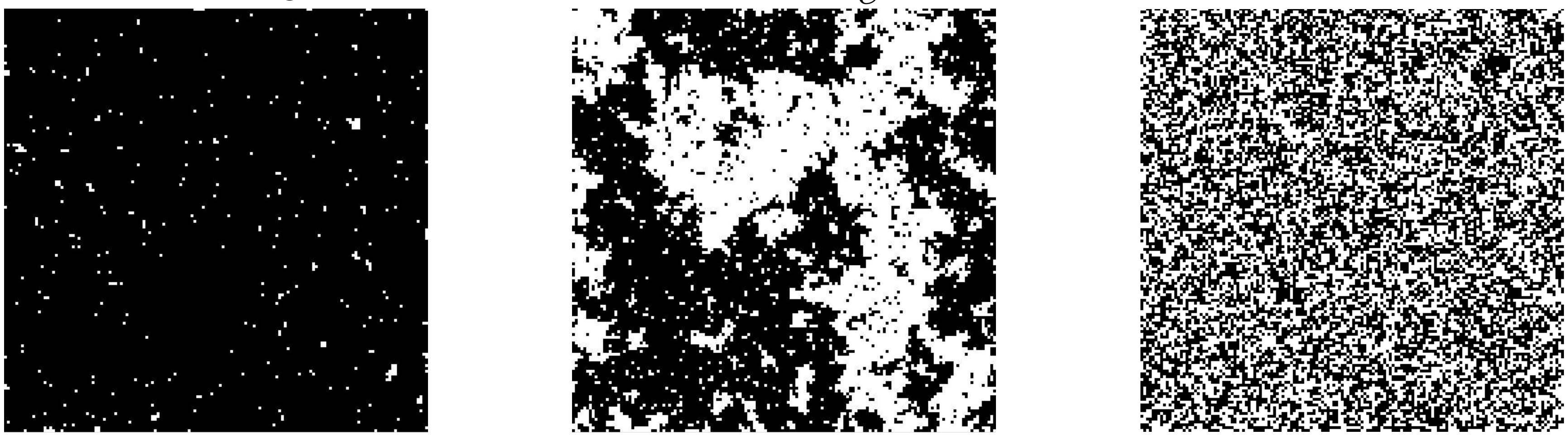
$T = \infty$: Entropy maximised: spins are randomly orientated: $m(\infty, 0) = 0$.

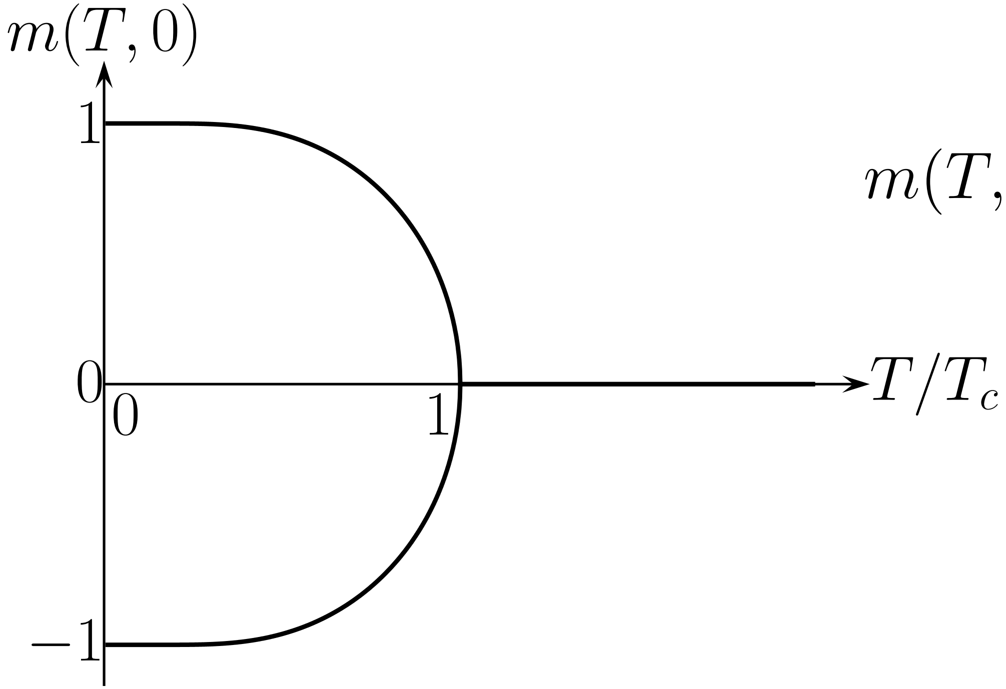
Con£gurations are self-similar. The correlation length $\xi(\infty, 0) = 0$.

$T = T_c$: $\langle E \rangle$ and $TS$ balanced. Spins are “undetermined”: $m(T_c, 0) = 0$.

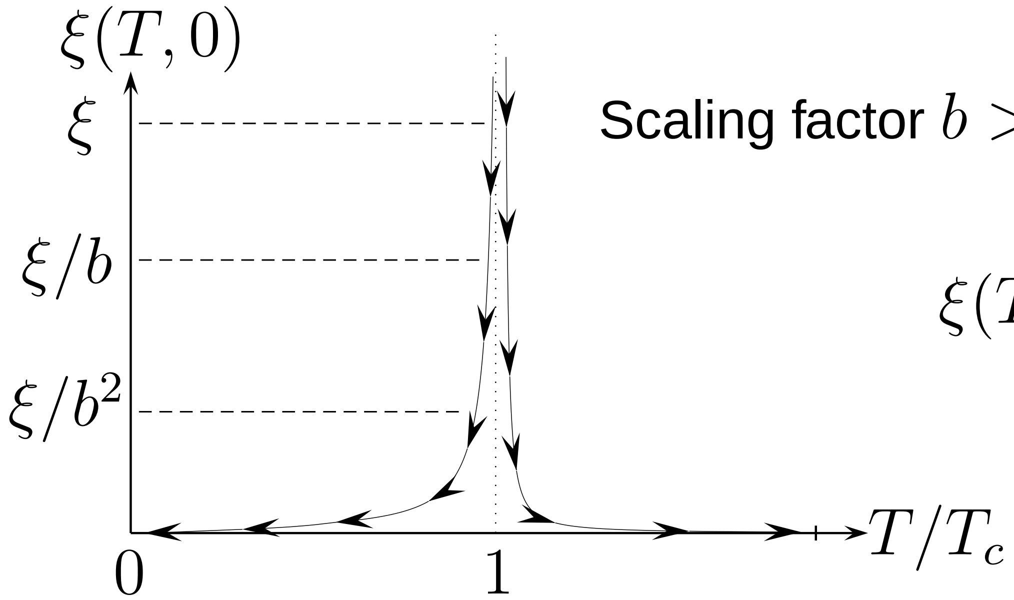
Con£gurations are self-similar. The correlation length $\xi(T_c, 0) = \infty$.

$T \to 0$ $\quad$ $T = T_c$ $\quad$ $T \to \infty$





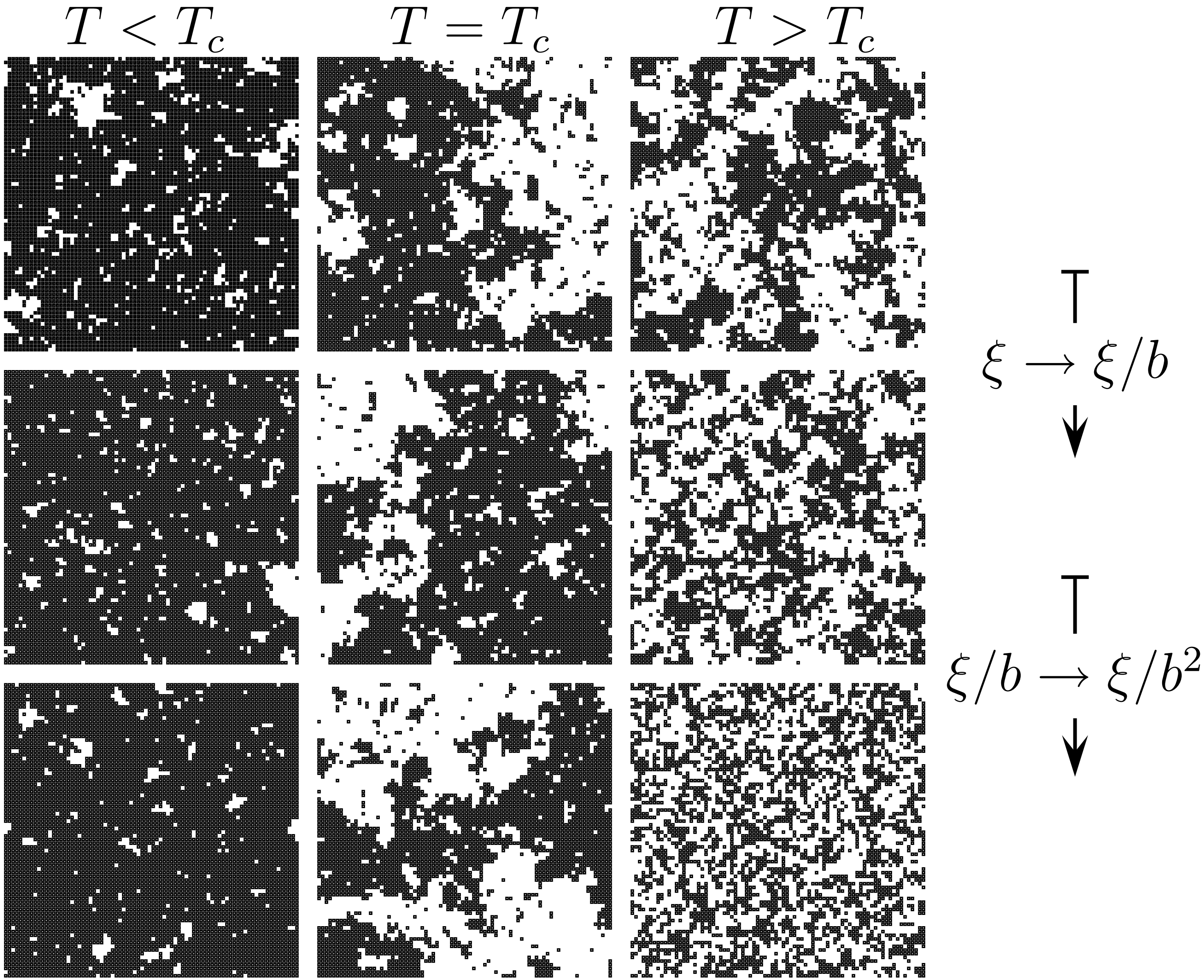

$$m(T,0) \propto \pm(T_c - T)^{\beta} \quad \text{for } T \to T_c^-.$$




Scaling factor $b > 1$.

$$\xi(T,0) \propto |T_c - T|^{-\nu} \quad \text{for } T \to T_c.$$

# Scale invariance and £xed points — Criticality: Ising Model




Fixed points $\xi = \xi/b$ $\quad \xi = 0 \quad \xi = \infty \quad \xi = 0$

# Warning

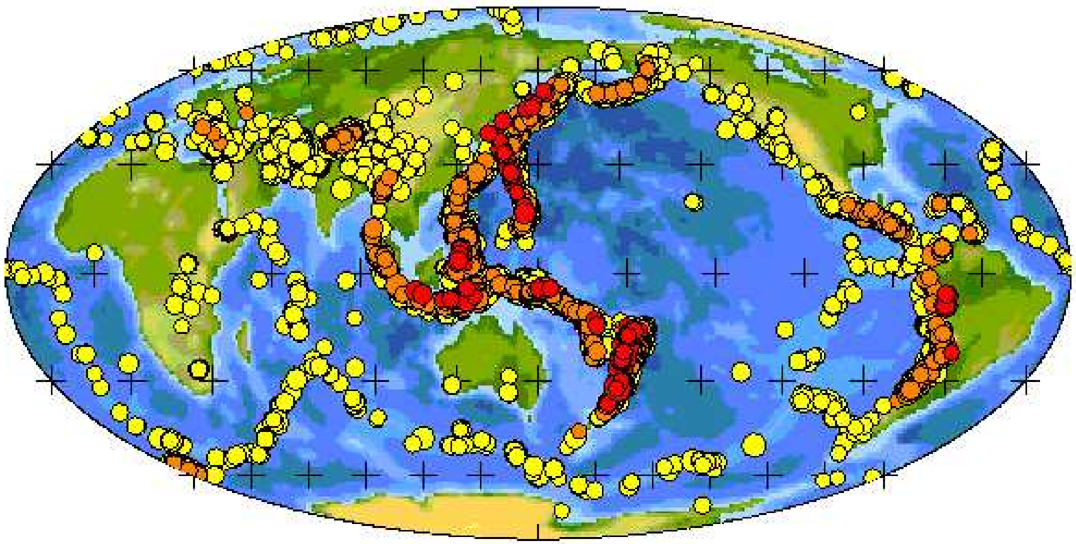


- No truly isolated natural systems exist.
- Most systems have a ¤ux of mass or energy passing though them.
- Most systems are in a non-equilibrium steady state.
- Take great care not to apply results from equilibrium systems outside their range of validity.

Palace Hotel, San Francisco, U.S.A.
5:12AM – 18 April, 1906.

World-wide occurrence of earthquakes.
Outline plate boundaries.

$1.1.1997 - 30.6.1997, M > 4$



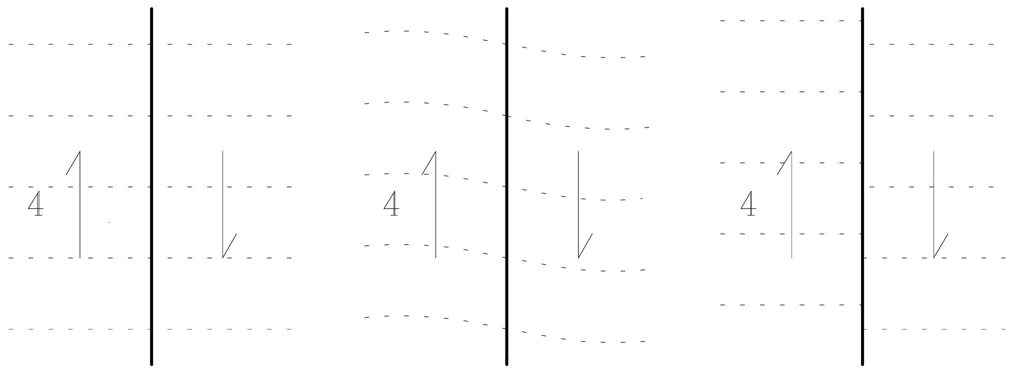
Earthquake catalogue

# Scale invariance: Gutenberg-Richter Law



At a fault, strain builds up slowly.

Energy released through earthquakes.



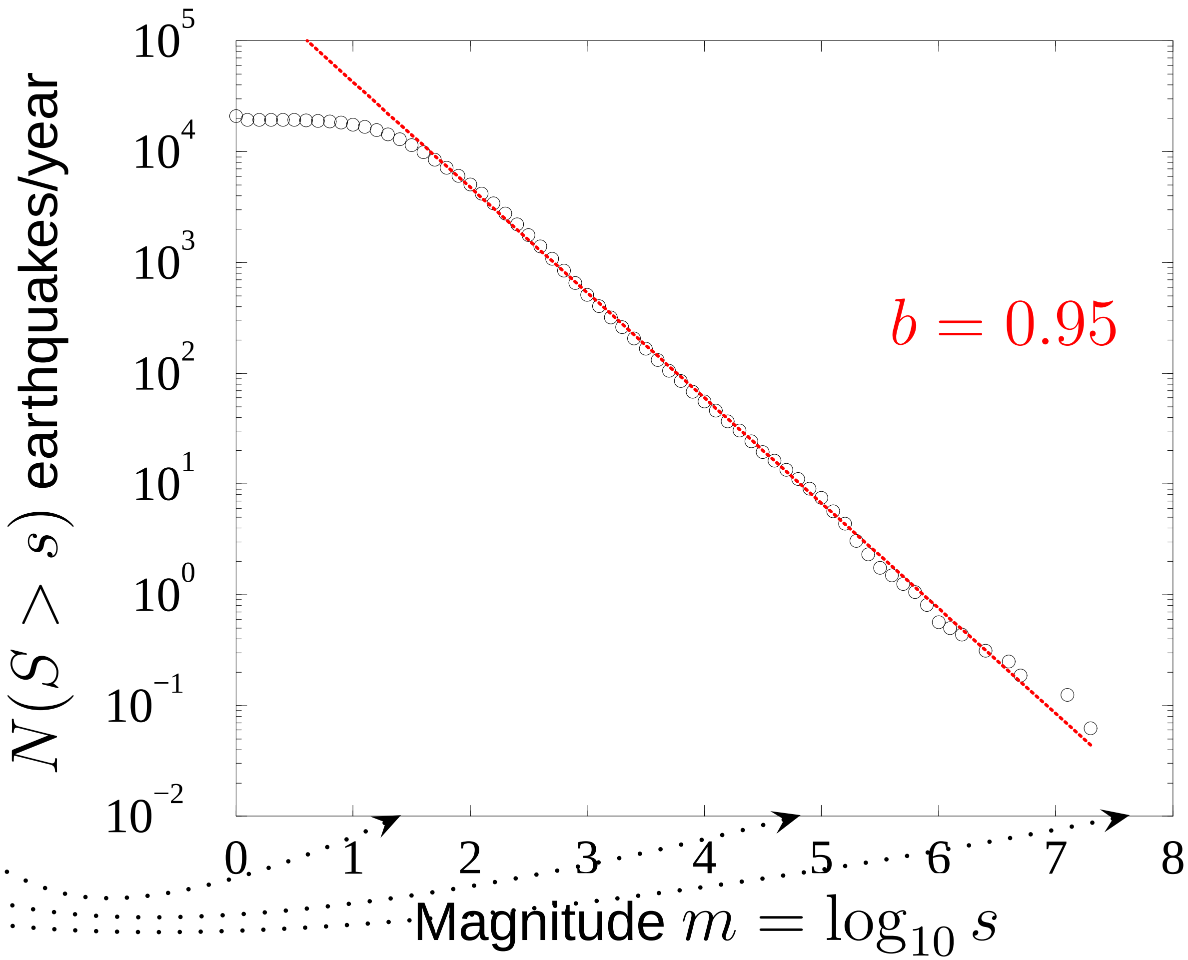

Gutenberg-Richter Law:

$N(S > s) \propto s^{-b}$.

Large truck passing by

Small atom bomb

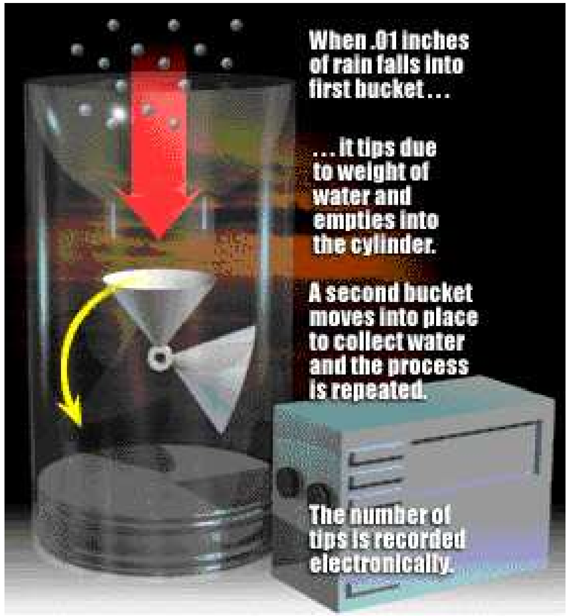
$100$ hydrogen bombs



Europe, August 2002.

Level of River Elbe $= 9.39$m.


Rain event over Grand Canyon dissipates energy in the atmosphere.




Standard rain gauge: Tipping bucket.

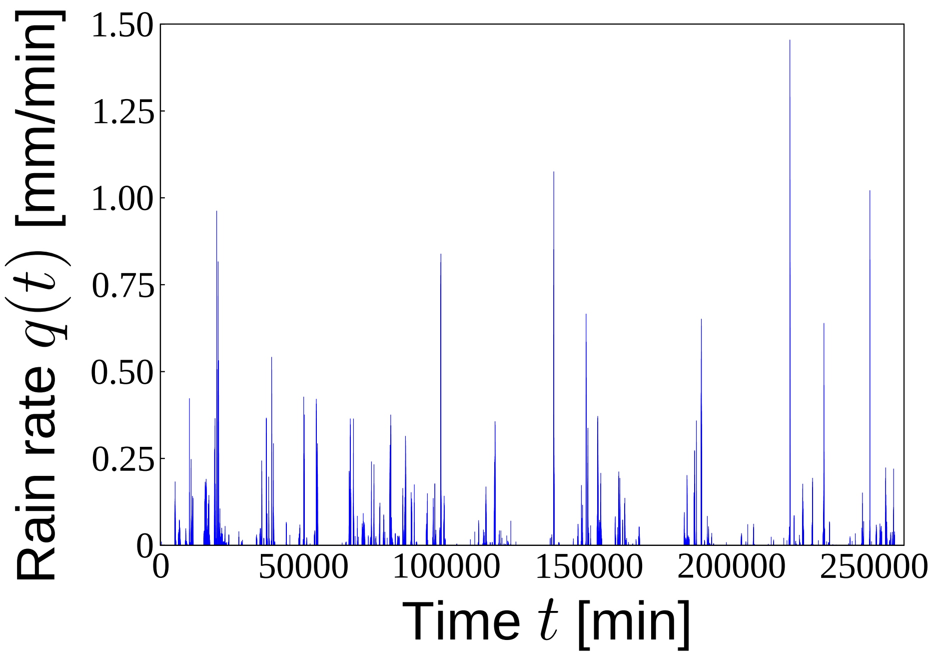
Resolution of rain rate $q_{min} = 0.25$ mm/h.

Temporal resolution $\Delta t = ?$ min.

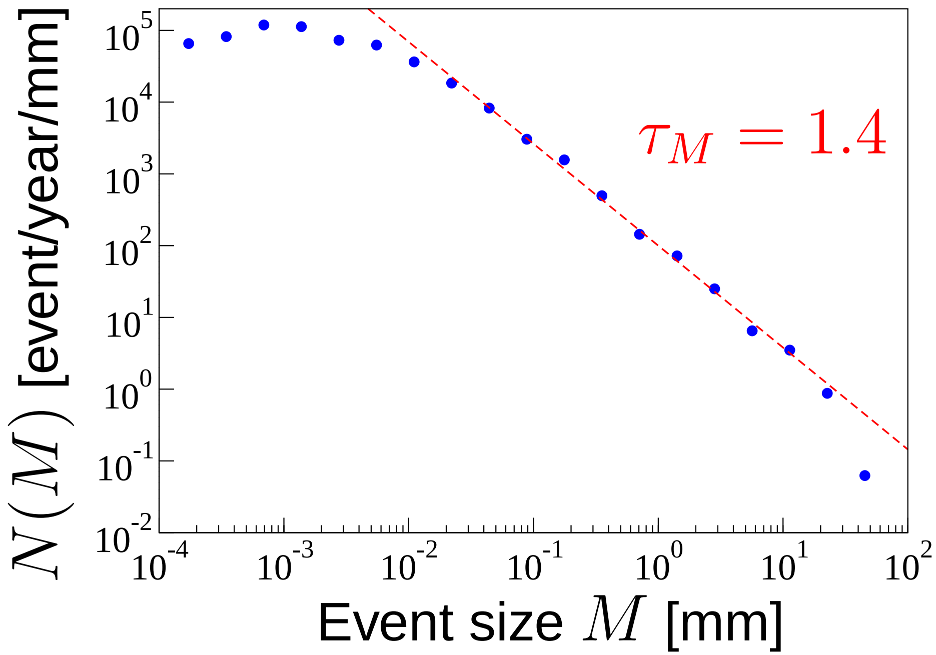
Radar: Resolution of rain rate $q_{min} = 0.005$ mm/h.

Temporal resolution $\Delta t = 1$ min.

A rain event is a sequence of successive non-zero rain rates. The event size $M = q(t+1) + \cdots + q(t+T)$. with event duration $T$.




Rain-equivalent of Gutenberg-Richter Law: $N(M) \propto M^{-\tau_M}$.

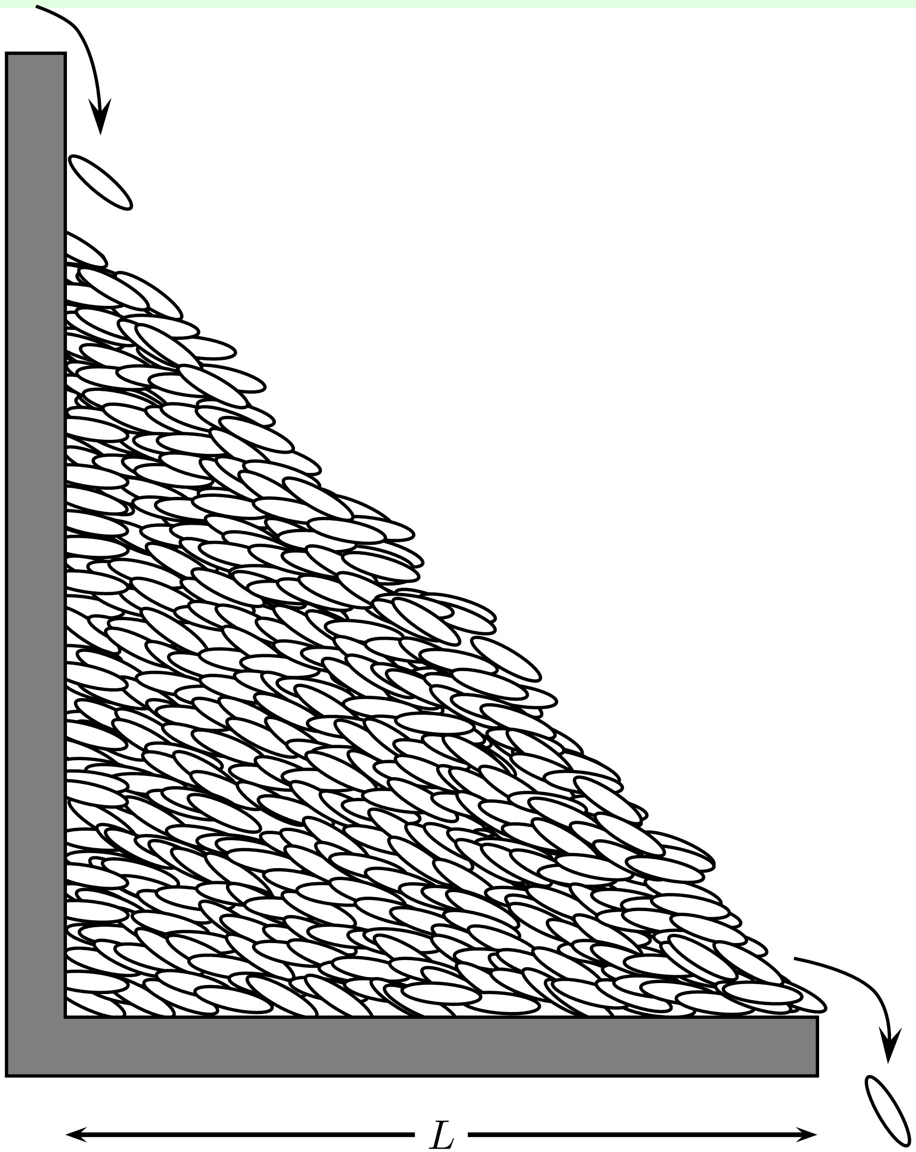


- Rain is a complicated spatio-temporal phenomenon.
- Trickles, drizzle, bursts, showers, downpours, and torrents.
- Identifying rain events as the basic entities reveals that
  - The frequency-event size distribution is scale free.

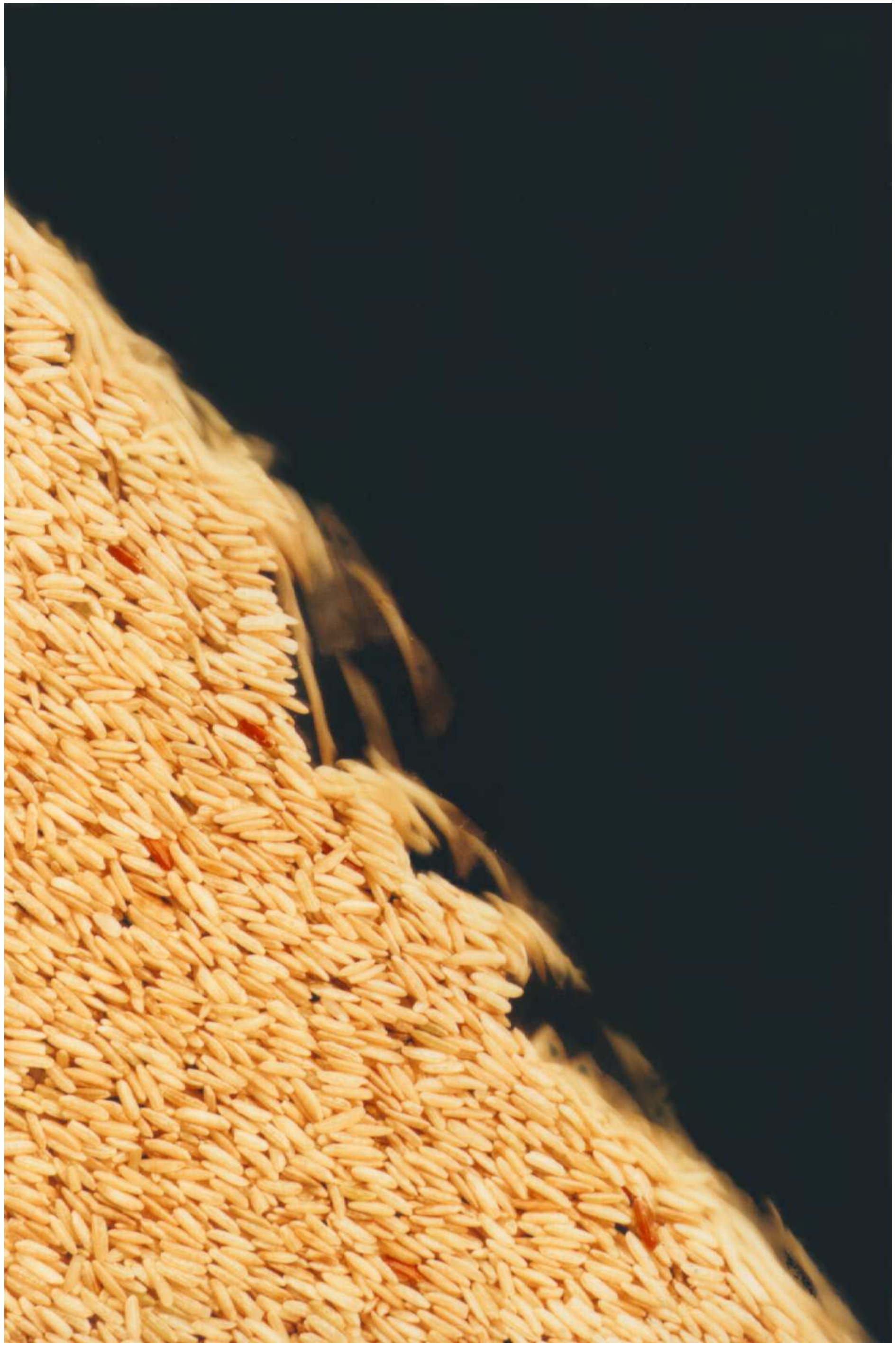
    This is the rain-equivalent of the Gutenberg-Richter law for earthquakes.
  - The frequency-drought duration distribution is scale free.

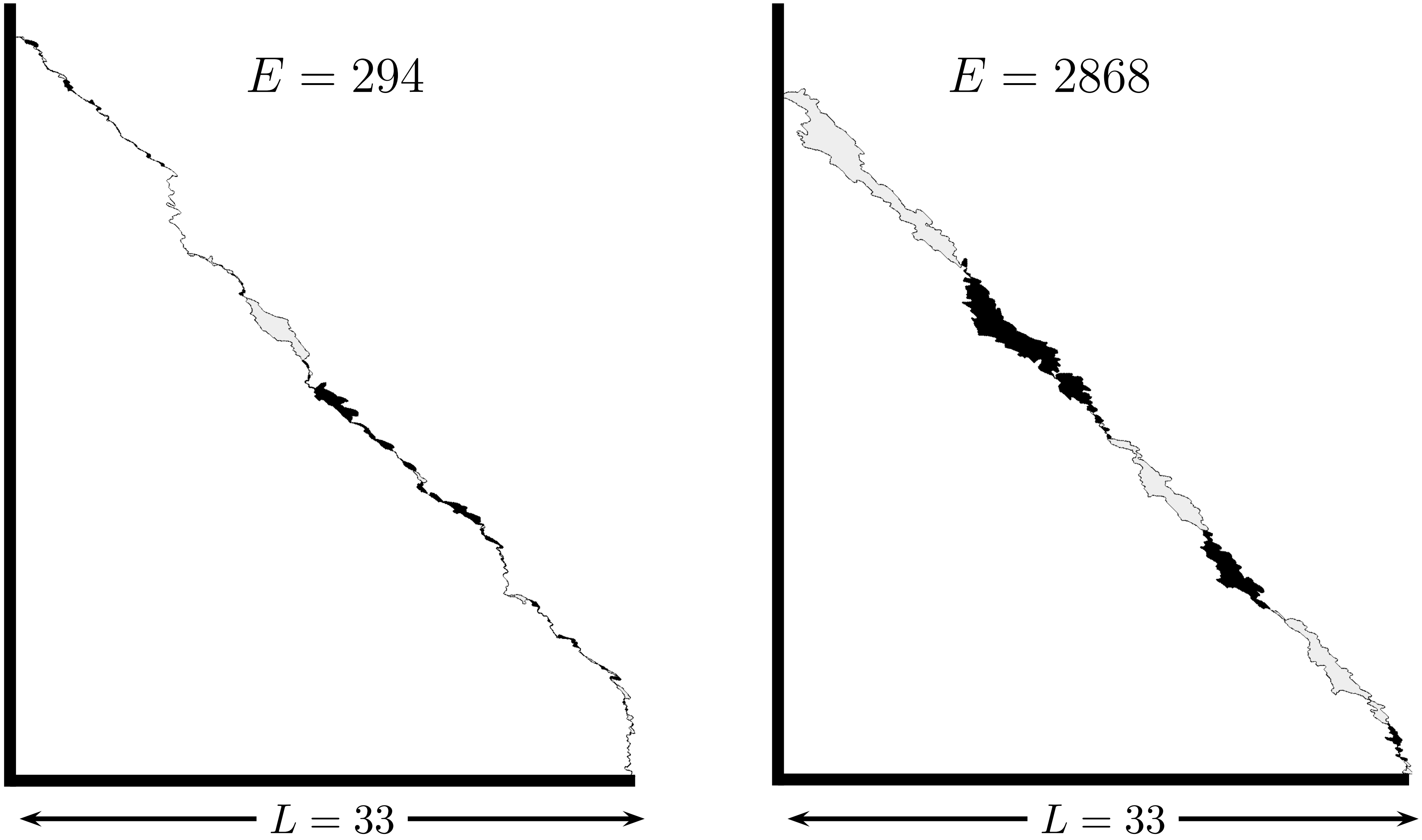
    This is the rain-equivalent of the Omori law for earthquakes.

Rain is “Earthquake in the Sky”.




- Reaches statistically stationary state where $\langle \text{influx} \rangle = \langle \text{outflux} \rangle$.
- Avalanches dissipate energy.

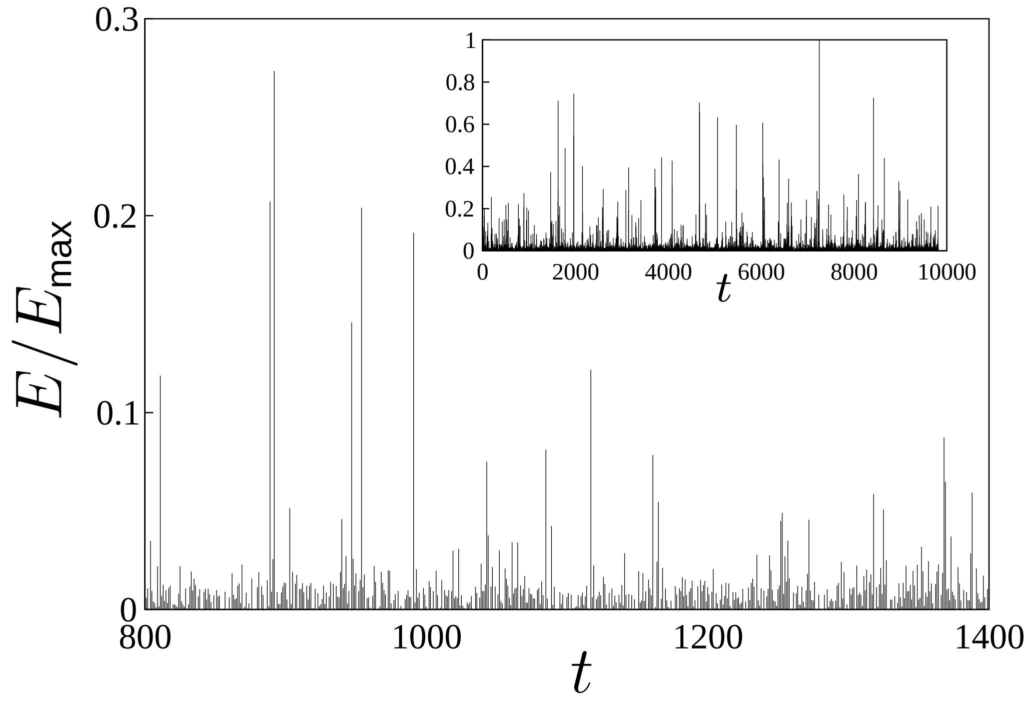




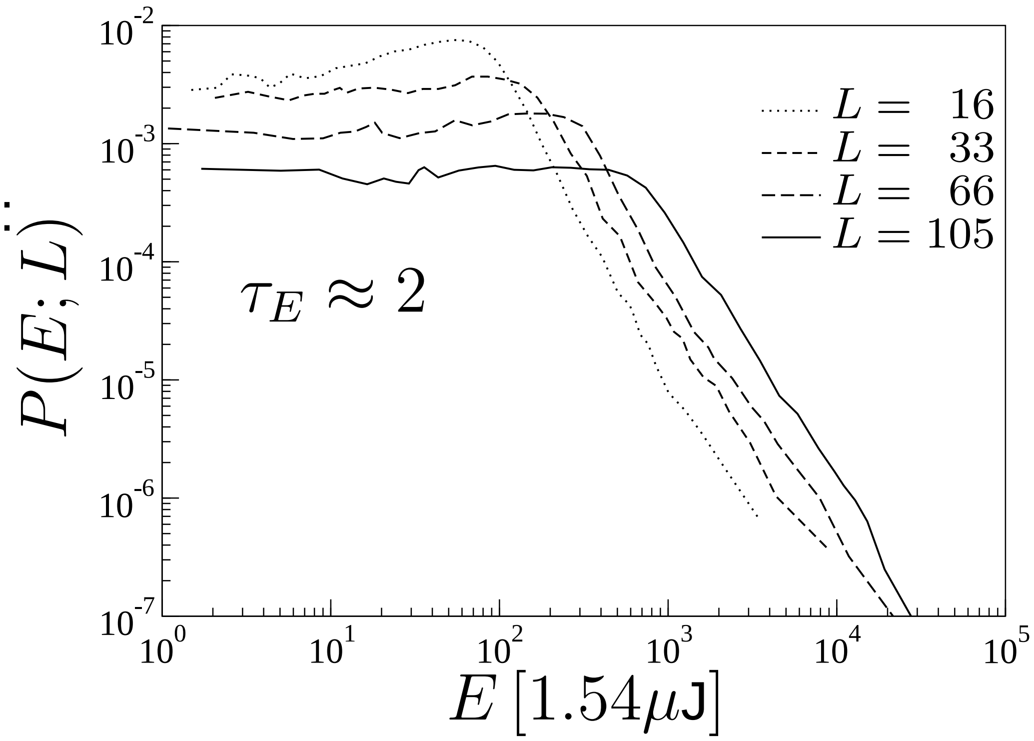

Energy dissipated by avalanche, $E$, measured in unit of $mg\delta = 1.54\mu J$.

Focus on avalanche-size probability density, $P(E; L)\, dE$, in system of size $L$.

Rescaled avalanche sizes, $E/E_{\mathrm{max}}$ as function of number of additions.




Avalanche-size probability densities: $P(E;L) \propto E^{-\tau_E}$ for $E \gg 1$.



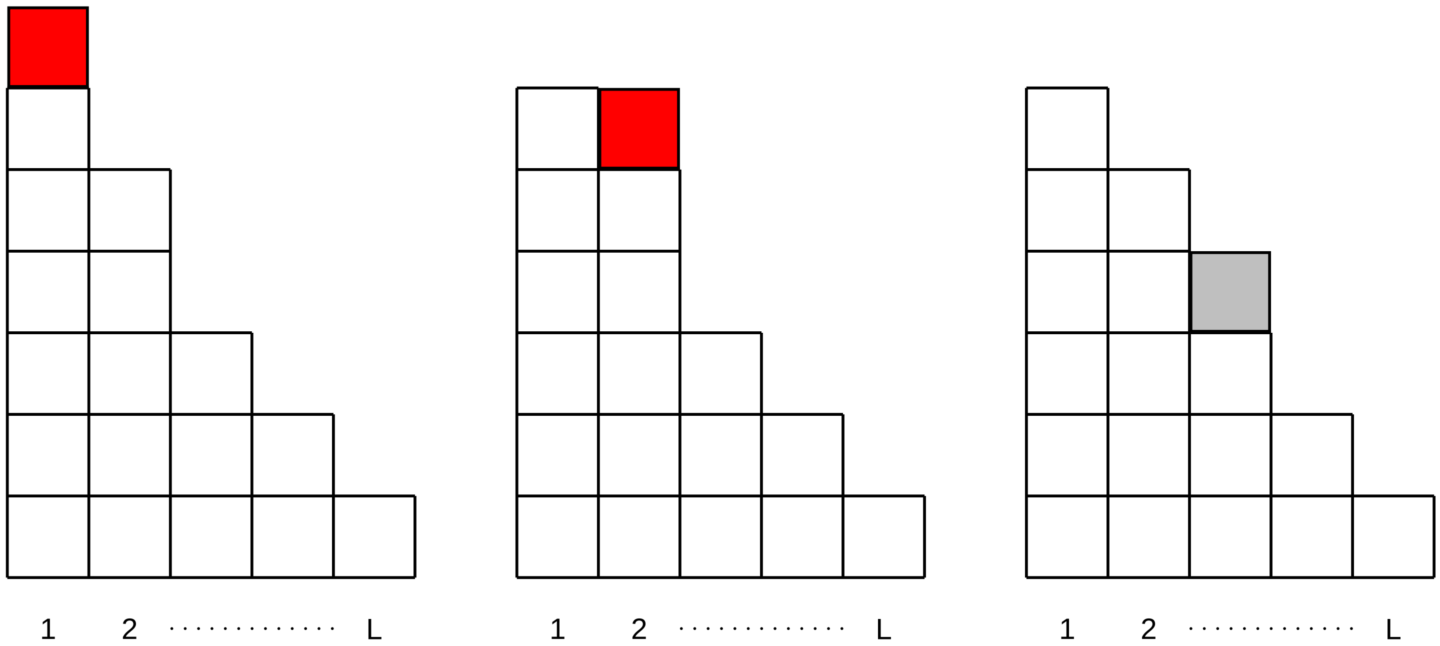
| System | Crust of Earth | Atmosphere | Granular pile |
|---|---|---|---|
| Energy source | Convection | Sun | Adding grains |
| Energy storage | Tension | Vapour | Potential |
| Threshold | Friction | Saturation | Friction |
| Relaxation | Earthquake | Rain event | Avalanche |

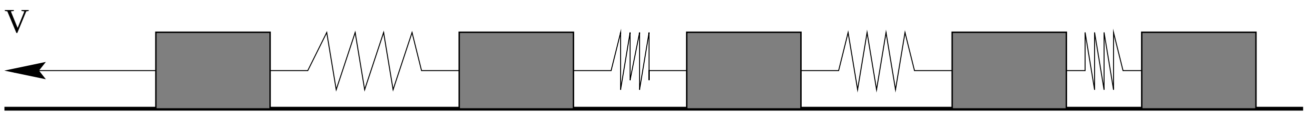
Common basis:

- Slowly driven non-equilibrium systems.
- Threshold dynamics.
- Relaxation event dissipates energy.
- Reaches statistically stationary state where $\langle \text{influx} \rangle = \langle \text{outflux} \rangle$.
- Relaxation events of all sizes up to a system dependent cutoff.

Lattice with $L$ sites, vertical wall at left boundary and open at right boundary.

- The height, $h_i$, is the number of grains at column $i$, with $h_{L+1} = 0$.
- The local slopes $z_i = h_i - h_{i+1}, \quad i = 1, \ldots, L$.




Train model of earthquakes.

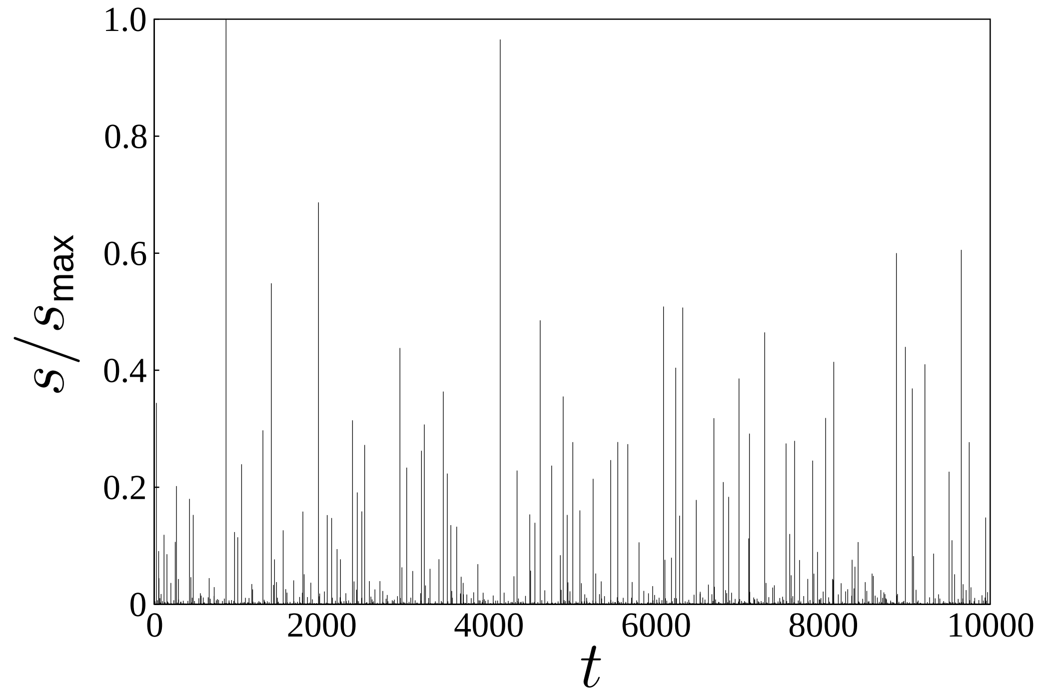
The algorithm for the Oslo rice-pile model:

1. Initialise the critical slopes $z_i^c \in \{1, 2\}$ and place the system in an arbitrary metastable state with $z_i \leq z_i^c$ for all $i$.
2. Add a grain at site $i = 1$: $z_1 \rightarrow z_1 + 1$.
3. If $z_i > z_i^c$, the site relaxes and

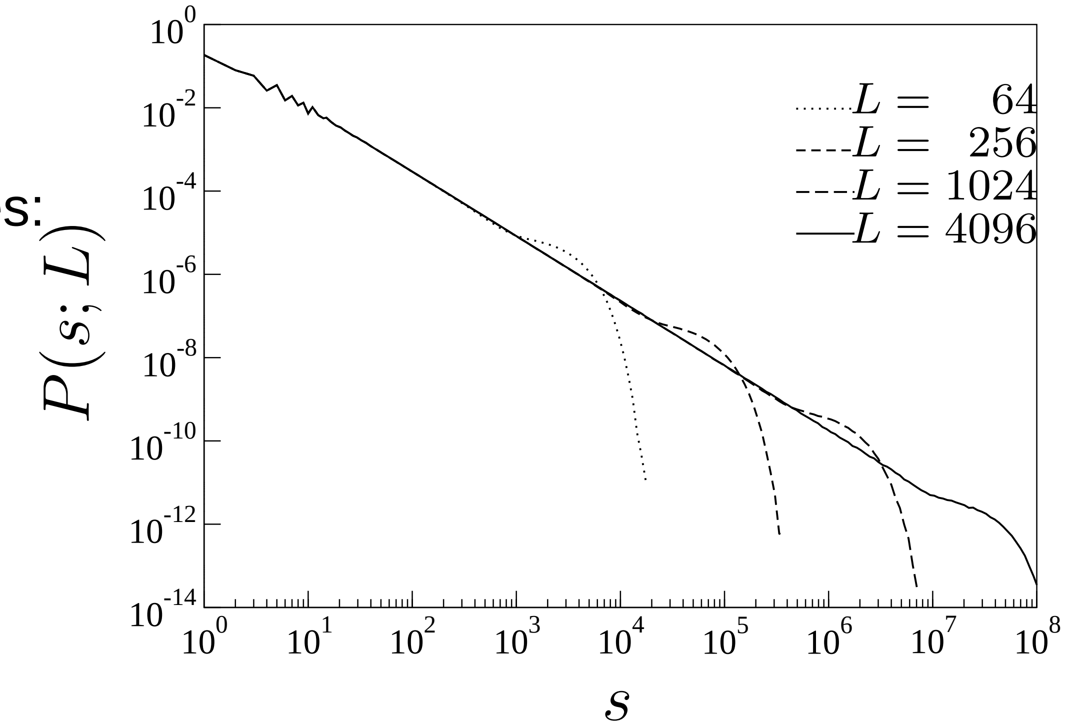
$$\begin{aligned} z_i &\rightarrow z_i - 2 \\ z_{i\pm1} &\rightarrow z_{i\pm1} + 1. \end{aligned}$$

   The critical slope $z_i^c$ is chosen randomly $z_i^c \in \{1, 2\}$
   A new metastable state is reached when $z_i \leq z_i^c$ for all $i$.
4. Proceed to 2 and reiterate.

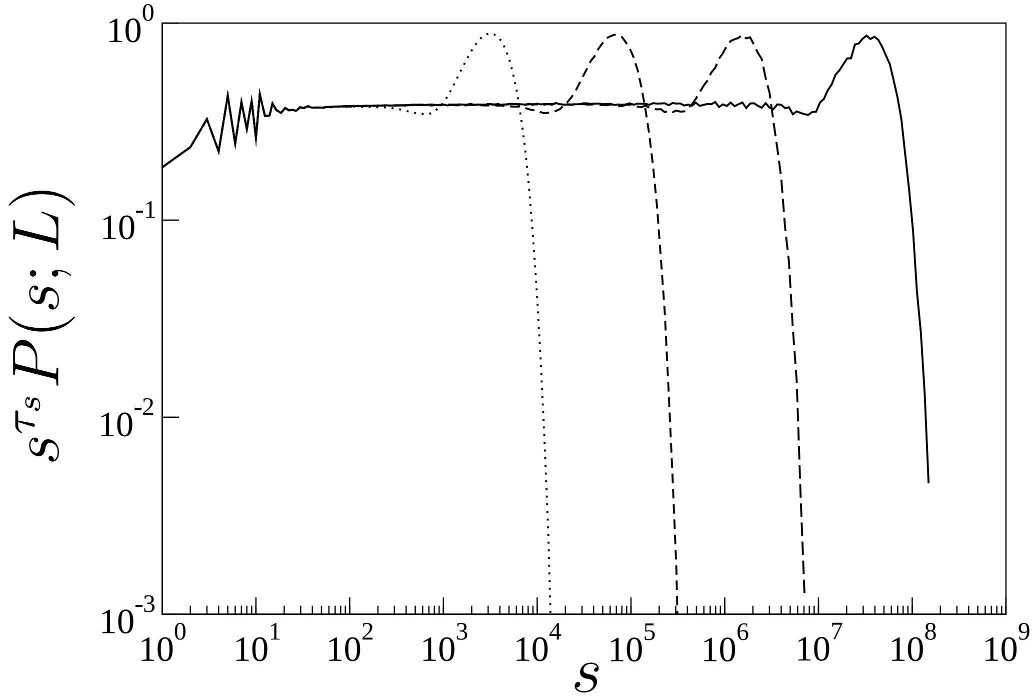
Rescaled avalanche sizes, $s/s_{\text{max}}$
as function of number of additions.




Avalanche-size probability densities:
$P(s;L) \propto s^{-\tau_s}\mathcal{G}(s/L^D)$.

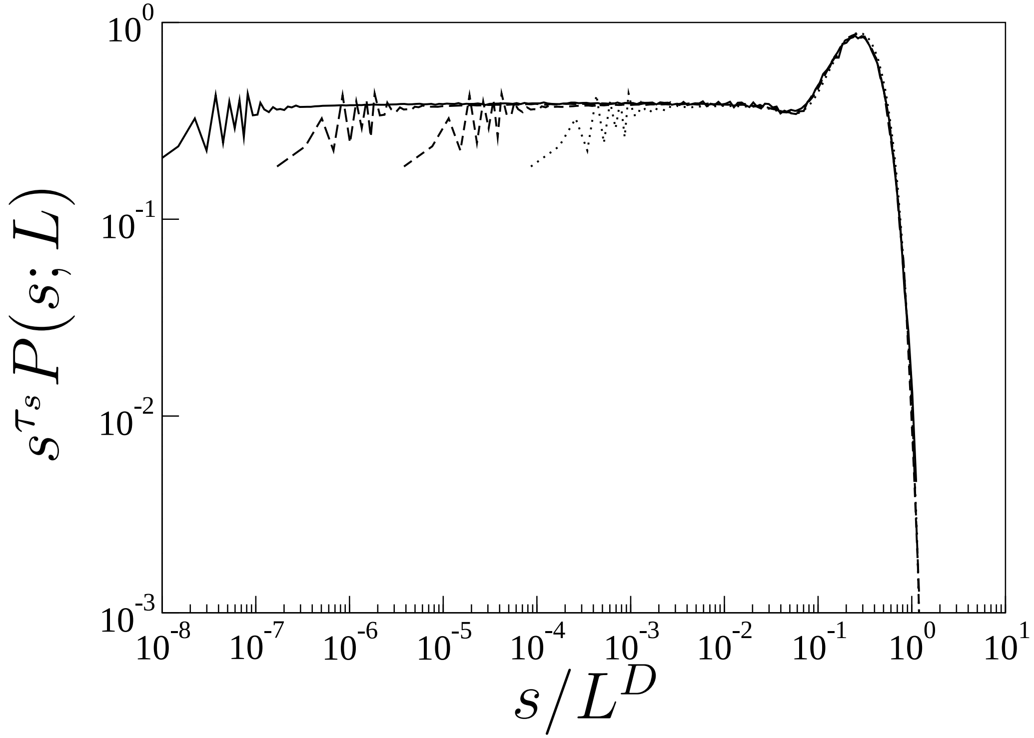


Transformed avalanche-size probability.




$s^{\tau_s} P(s;L) \propto \mathcal{G}(s/L^D)$.

Data collapse obtained using exponents $\tau_s = 1.55, D = 2.25$.



# Conclusion



- The susceptibility is the average avalanche size: $\langle s \rangle = L$.
- The slowly driven pile organises itself, without any external £ne-tuning of control parameters, into a highly susceptible state where the susceptibility diverges with system size.
- The Oslo rice-pile model displays self-organised criticality.
- The Oslo model is the "Ising Model" for self-organised criticality.

Per Bak.
Leon Danon, University of Barcelona, Spain.
Tim Scanlon, Imperial.
Vidar Frette, Haugesund Højskole, Norway.
Anders Malthe-Sørenssen, University of Oslo, Norway.
Jens Feder, PGP, University of Oslo, Norway.
Torstein Jøssang, PGP, University of Oslo, Norway.
Paul Meakin, PGP, University of Oslo, Norway.
Alvaro Corral, Universitat Autonoma de Barcelona, Spain.
Gunnar Pruessner, Imperial.
Matthew Stapleton, Imperial.
Nicholas Moloney, Elte University, Budapest, Hungary.
Ole Peters, Santa Fe Institute and Los Alamos Natinal Laboratory, U.S.A.
Christopher Hertlein, Germany.

Papers/animations available via: www.cmth.ph.ic.ac.uk/people/k.christensen/