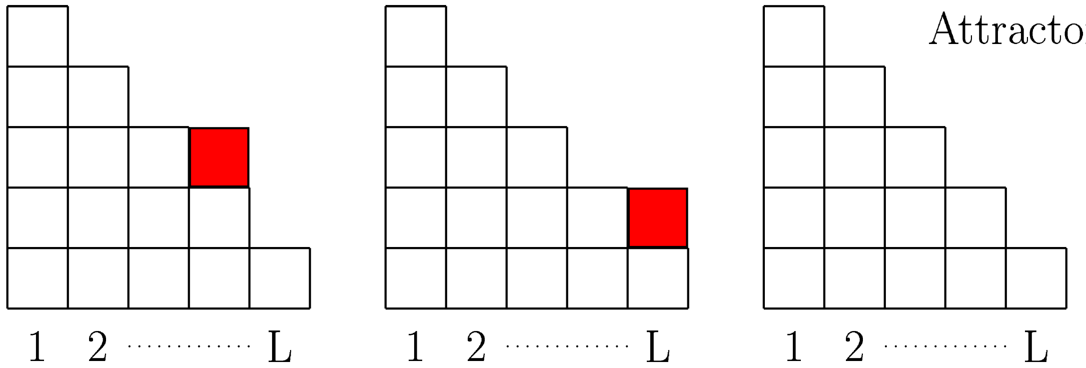
Science of Complexity – Summary of L22

Bak-Tang-Wiesenfeld $d = 1$ sandpile model.



Attractor (fixed point).

Dynamical variable: local slopes $z_i = h_i - h_{i+1}$. Dynamics:

1. Initialisation: Place pile in stable configuration $\{z_i\}_{i=1}^{L}$, with $z_i \leq z^{\text{th}}$.
2. Drive: Add a grain at site i:
$$z_i \to z_i + 1$$
$$z_{i-1} \to z_{i-1} - 1$$
3. If $z_i > z^{\text{th}}$, the site topples (relaxes) and
$$z_i \to z_i - 2$$
$$z_{i\pm1} \to z_{i\pm1} + 1.$$
   A new stable configuration is reached when $z_i \leq z^{\text{th}}$ for all i.
4. Proceed to 2 and reiterate.

A stable configuration has $z_i \leq z^{\text{th}}$ for all i. Adding a grain to a stable configuration $\mathcal{S}_j$, it evolves into a new stable configuration $\mathcal{S}_j \mapsto \mathcal{S}_{j+1}$. Stable configurations are either transient configurations $\mathcal{T}$ or recurrent configurations $\mathcal{R}$. Symbolically,

$$\mathcal{T}_1 \mapsto \mathcal{T}_2 \mapsto \cdots \mapsto \mathcal{T}_n \mapsto \underbrace{\mathcal{R}_1 \mapsto \cdots \mapsto \mathcal{R}_{j-1} \mapsto \mathcal{R}_j \mapsto \mathcal{R}_{j+1} \mapsto \cdots}_{\textit{attractor}}$$

Size s of avalanches = total no. of relaxations in the transition $\mathcal{S}_j \mapsto \mathcal{S}_{j+1}$. When the $d = 1$ BTW model is in the attractor, the avalanche size probability:

$$P(s, L) = \begin{cases} 1/L & \text{if } 1 \leq s \leq L \\ 0 & \text{otherwise} \end{cases} = s^{-1}\frac{s}{L}\Theta(1 - \frac{s}{L}) = s^{-1}\mathcal{G}_{\text{1d}}^{\text{BTW}}(s/L)$$

Generally, we would expect the avalanche size probability

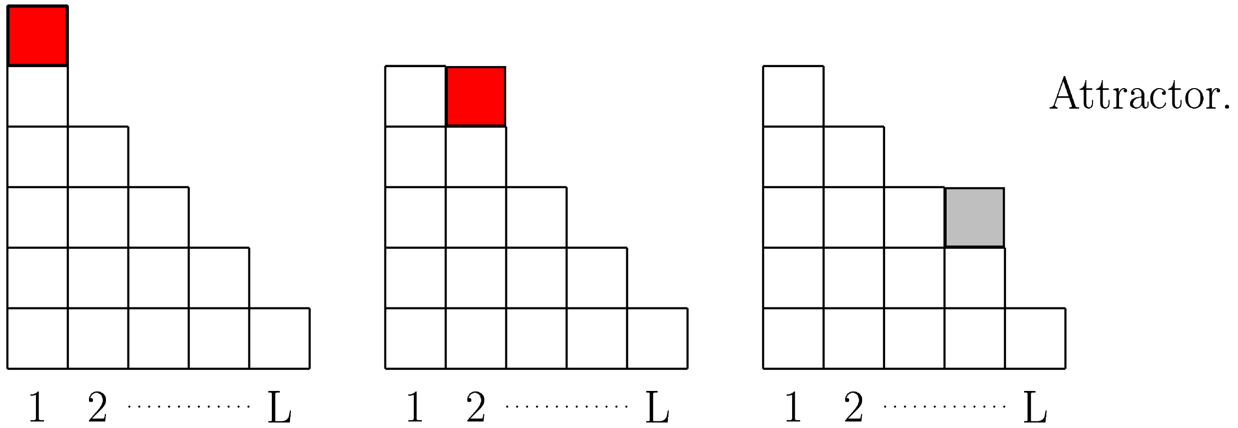
$$P(s, L) \propto s^{-\tau}\mathcal{G}(s/L^D) \quad \text{for } s \gg 1, L \gg 1,$$

where τ, D are critical expoents and $\mathcal{G}$ a scaling function where

$$\mathcal{G}(x) = \begin{cases} \mathcal{G}(0) + \mathcal{G}'(0)x + \cdots & \text{for } x \ll 1 \\ \text{decays rapidly} & \text{for } x \gg 1. \end{cases}$$

Science of Complexity – Summary of L23

Oslo ricepile model in d=1.



Dynamical variable: local slope $z_i = h_i - h_{i+1}$. Threshold slope $z_i^{\rm th} \in \{1, 2\}$.

1. Initialisation: Place pile in stable configuration $\{z_i\}_{i=1}^L$, with $z_i \le z_i^{\rm th}$.
2. Drive: Add a grain at site $i = 1$: $z_1 \to z_1 + 1$.
3. Relaxation: If $z_i > z_i^{\rm th}$, the site topples (relaxes) and $\quad z_i \to z_i - 2$
   $$z_{i\pm 1} \to z_{i\pm 1} + 1.$$
   Threshold $z_i^{\rm th}$ is chosen anew. A new stable configuration is reached when $z_i \le z_i^{\rm th}$ for all i.
4. Proceed to 2 and reiterate.

$$\mathcal{T}_1 \mapsto \mathcal{T}_2 \mapsto \cdots \mapsto \mathcal{T}_n \mapsto \underbrace{\mathcal{R}_1 \mapsto \cdots \mapsto \mathcal{R}_{j-1} \mapsto \mathcal{R}_j \mapsto \mathcal{R}_{j+1} \mapsto \cdots}_{attractor}$$

In the attractor, the avalanche size probability

$$P(s, L) = s^{-\tau} \mathcal{G}_{\rm 1d}^{\rm Oslo}(s/L^D) \quad \text{for } s \gg 1, L \gg 1.$$

where τ, D are critical exponents and $\mathcal{G}_{\rm 1d}^{\rm Oslo}$ a scaling function.
The kth moment of the avalanche size density

$$\langle s^k \rangle = \sum_{s=1}^{\infty} s^k P(s, L) \approx L^{D(1+k-\tau)} \int_{1/L^D}^{\infty} \mathcal{G}(\tilde{s})\, d\tilde{s} \propto L^{D(1+k-\tau)} \quad \text{for } L \to \infty.$$

For the Oslo model, the average avalanche size (susceptibility) $\langle s \rangle = L$, so

$$D(2 - \tau) = 1.$$

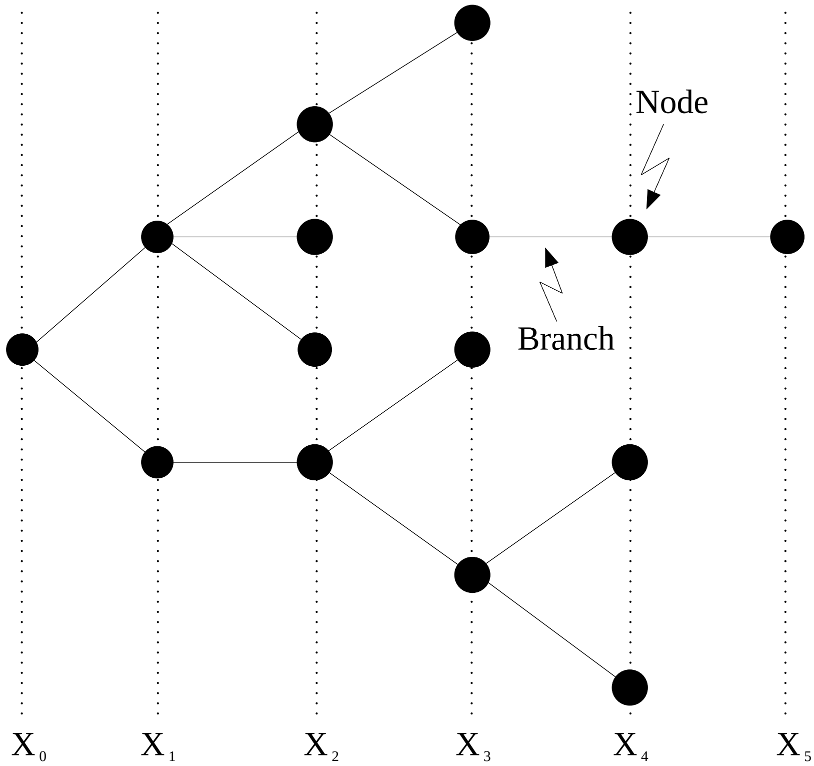
Numerical results

$$D = 2.25(1) \stackrel{?}{=} \frac{9}{4} \quad \text{and} \quad \tau = 1.55(1) \stackrel{?}{=} \frac{14}{9}.$$

Science of Complexity – Summary of L24

Avalanche dynamics.

Branching process (BP):



X_j = is the number of nodes in generation j:

Probability p_b that a node has b branches. **The branching ratio**

$$\langle b \rangle = \sum_{b=0}^{\infty} b p_b = \begin{cases} < 1 & \text{sub-critical} \\ = 1 & \text{critical} \\ > 1 & \text{super-critical.} \end{cases}$$

The BP process displays a phase transition at $\langle b \rangle = 1$. Total number of nodes in a tree $s = \sum_{j=0}^{\infty} X_j$ – avalanche size. For an uncorrelated BP

$$\begin{aligned} P(s, \langle b \rangle) &\propto s^{-\tau} \exp(-s/s_\xi) && \text{for } s \gg 1 \\ s_\xi &\propto |1 - \langle b \rangle|^{-1/\sigma} && \text{for } \langle b \rangle \to 1 \end{aligned}$$

with $\tau = 3/2, \sigma = 1/2$, implying an **average tree (avalanche) size**

$$\langle s \rangle \propto s_\xi^{2-\tau} \propto |1 - \langle b \rangle|^{-(2-\tau)/\sigma} \Leftrightarrow \langle b \rangle = 1 - \frac{1}{\langle s \rangle}.$$

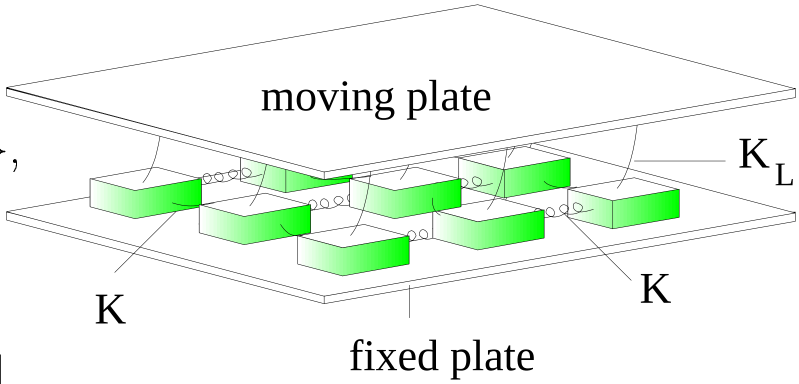
Avalanches in SOC systems can be view as a correlated branching process. A given avalanche k has branching ratio $b_k = (s_k - 1)/s_k$ so

$$\langle b \rangle = \frac{\sum_k s_k b_k}{\sum_k s_k} = 1 - \frac{1}{\langle s \rangle}$$

generalising the result for uncorrelated BP. The SOC systems **self-organise** into states with $\langle b \rangle \to 1$ as $\langle s \rangle \to \infty$. **No fine-tuning** of a control parameter is needed. Nonconservative sanpile models have $\langle b \rangle < 1$. They are sub-critical.

Science of Complexity – Summary of L25

Burridge-Knopoff spring-block model of fault.



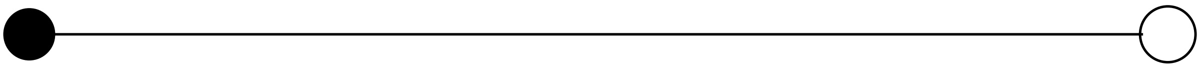
1. Random initial condition $\{F_i\}$, with $F_i < F_c = 1$.
2. Increase strain uniformly.
3. If $F_i \geq F_c$, the block slips and

$$
\begin{aligned}
F_{nn} &\rightarrow F_{nn} + \alpha F_i, \quad \alpha = \frac{K}{4K + K_L} \\
F_i &\rightarrow 0
\end{aligned}
$$

   A new metastable state is reached when $F_i < F_c$ for all i.

4. Proceed to 2 and reiterate.

Dissipation when bulk site slips $\Delta F = F_i^{toppling} - 4\alpha F_i^{topling} = (1 - 4\alpha) F_i^{toppling}$.



$\alpha = 0$
Independent oscillators.

$\alpha = \frac{1}{4}$.
Conservative model

Critical model for $\alpha_c \leq \alpha \leq \frac{1}{4}$ where $\alpha_c =?$. Recent results indicate that the density of avalanche sizes

$$
P(s, L) \propto s^{-\tau} \mathcal{G}(s/L^D), \quad \tau \approx \begin{cases} 1.25 & \text{for } \alpha = \frac{1}{4} \\ 1.8 & \text{for } \alpha_c \leq \alpha < \frac{1}{4}. \end{cases}
$$

Let $\langle F_{max} \rangle = \langle \mathrm{Max} F_i \rangle$ when system is in a metastable state. The $\langle \mathrm{Influx} \rangle = \langle \mathrm{Outflux} \rangle$ implies $L^2 (F_c - \langle F_{max} \rangle) = \langle s \rangle (1 - 4\alpha) \langle F_i^{toppling} \rangle$ that is

$$
\langle s \rangle = \frac{L^2 (F_c - \langle F_{max} \rangle)}{(1 - 4\alpha) \langle F_i^{toppling} \rangle} \rightarrow \infty \quad \text{for } L \rightarrow \infty
$$

if

$$
\alpha \rightarrow 1/4 \quad \text{or} \quad (F_c - \langle F_m ax \rangle) \propto L^{-q}, q < 2.
$$

The behaviour of the Olami-Feder-Christensen model is still under scrutiny.