PHYSICAL REVIEW E VOLUME 48, NUMBER 5 NOVEMBER 1993

Sandpile models with and without an underlying spatial structure

Kim Christensen
Department of Physics, University of Oslo, P.O. Box 1048, Blindern, N-0316 Oslo 3, Norway

Zeev Olami
Department of Physics, Brookhaven National Laboratory, Upton, New York 11973
(Received 1 February 1993; revised manuscript received 2 August 1993)

We present a simple mean-field model for the sandpile model introduced by Bak, Tang, and Wiesenfeld (BTW) [Phys. Rev. Lett. **59**, 381 (1987)]. In the mean-field model we are able to pinpoint the process of self-organization as well as the emerging scale invariance displayed as a power-law distribution of avalanche sizes. We discuss the BTW sandpile model on a lattice and show that the dynamical behavior can be expressed as a transport problem. This implies that the average avalanche size scales with the system size, and additional heuristic arguments related to the transport properties more than indicate the origin of the power-law behavior. We review recent work in which scaling relations and additional constraints between the various critical exponents are addressed. We demonstrate that some of the proposed relations are inconsistent. We present a coherent "theory" in which the scaling relations along with additional constraints leave only one exponent unknown.

PACS number(s): 05.40.+j, 05.70.Jk, 05.70.Ln

I. INTRODUCTION

A few years ago Bak, Tang, and Wiesenfeld (BTW) [1] suggested that the frequent occurrence of fractal structures [2,3] is the generic spatial characteristic of a dynamical critical state into which dynamical systems with many spatial degrees of freedom evolve naturally. Unlike phase transitions in an equilibrium system, a driven dissipative dynamical many-body system reaches the critical state without the need to adjust the system parameters, i.e., the critical state studied by BTW is an attractor of the dynamics. Therefore, the critical state is usually described as being self-organized.

This idea assumes increasing importance in the new era of physics, where the focus is on "developing complexity out of simplicity" in contrast with the attempt "to reduce complexity to simplicity" to use the words of Anderson [4]. The fractal growth phenomenon such as diffusion-limited aggregation (in which particles perform a random walk until they reach the growing cluster where they come to rest) is an example in which simple local rules lead to a great variety of physical phenomena including scale invariance [3,5]. And indeed, the relationship between the simple underlying rules governing the dynamics of extended physical systems and the emerging complex structures is an intriguing problem [6].

Often a sandpile is used as a paradigm of an extended many-body system displaying self-organized criticality. As an example, take a square table and a large bucket of sand. We begin sprinkling grains of sand on the table, one grain at a time. We drop the grain on a randomly chosen place on the table and repeat the act when all motion has terminated. In the beginning the grains just fall on the table in no particular pattern. After a while, small local avalanches are created in order to decrease the local slopes whenever they become too steep, and eventually we end up with only one big sandpile. At some point (the transient time) this pile ceases to grow. The (global) average slope has reached a steady state corresponding to the angle of repose which the sandpile cannot exceed no matter how much sand we add. The pile has reached a statistically stationary state and additional grains of sand will ultimately fall off the pile. The avalanches induce the transport of sand which is clearly necessary in order to relax the sandpile.

In order to examine the phenomenon of self-organized criticality, BTW introduced a cellular automaton which involves discrete space coordinates [1]. The dynamical rules in the BTW model—also known as the sandpile model—at least intuitively resemble the dynamics of a sandpile: A signal is transmitted from a local site to its nearest neighbors the moment a dynamical integer variable (local slope) exceeds a critical value (the angle of repose).

By simulating this model BTW showed that the system does indeed drive itself to a statistically stationary state, characterized by the distributions of avalanche lifetimes and avalanche sizes which exhibit power-law behavior limited only by the size of the system. Since the system evolves into a stationary state without any characteristic time or length scales, it is in this sense critical. The generic universality of the model stems from the fact that the systems reach the critical attractor without the need to adjust the system parameters. Also, the systems adjust themselves to different environments. In the language of sandpiles, it does not affect the criticality of the final stationary state if we use wet sand instead of dry sand [1].

We stress that the complexity of these models is not a result of complex local rules; the complexity emerges as a result of the continued local interaction between all parts in the extended system. Such examples of dynamical systems which generate complex fractal structures might provide the explanation of the common appearance of fractal structures in nature. Note that the idea of self-organized criticality (SOC) refers to an extended system with *many* degrees of freedom. It complements in some sense the concept of "chaos" in which simple systems with a *few* degrees of freedom display quite complex behavior.

The tale of the sandpile provides an intuitive picture of the basic concepts of self-organized criticality. Several authors have addressed the interesting question as to whether or not real sandpiles display SOC [7–11]. Since the purpose of the present paper is not to discuss the behavior of real sandpiles in detail but rather the behavior of the cellular-automaton model associated with this, we only give a brief summary of the experiments on real sandpiles and refer the reader to the references herein.

Two distinct types of experiments have been performed: (a) rotating at a low constant velocity a semicylindrical drum partially filled with sand [7–9], and (b) dropping at a low rate individual grains on a conical sandpile resting on a circular platform [10,11].

Using method (a) Jaeger, Liu, and Nagel [7,8] placed a pair of capacitor plates below the rim of the rotating drum in order to monitor the flow of sand over the rim. They found only large avalanches occurring (nearly) periodic in time. It is argued that when the "sandpile" reaches an upper maximum angle of stability θ_m a large avalanche occurs and the slope of the pile is reduced to θ_r, the angle of repose. When the system once again reaches the maximum angle of stability θ_m a new large avalanche is initiated. The existence of $\theta_m > \theta_r$ implies hysteresis associated with a first-order transition: When $\theta_r < \theta < \theta_m$ the sandpile can either be stationary (building up) or flowing (relaxing). No indication of SOC behavior (second-order transition) was observed.

Experiments of type (b) were performed by Held *et al.* [10] and Rosendahl, Vekić, and Kelley [11], who used a balance to measure the mass fluctuations in the sandpile, i.e., the distribution in the avalanche sizes of the flow of sand over the rim. Held *et al.* observed that the distribution of sand flowing over the rim did obey a power law showing finite-size scaling, but *only* when the pile was sufficiently small. In larger sandpiles the behavior was similar to that reported by Jaeger, Liu, and Nagel, i.e., relaxational oscillations. It was suggested that SOC might be interpreted as a finite-size effect, since only small sandpiles display SOC in the experiment by Held *et al.* An explanation for the crossover from the apparent SOC behavior in small sandpiles to the oscillatory behavior in large sandpiles was offered by Liu, Jaeger, and Nagel [12]: If the length of the sandpile L is too small, then the addition of a single grain of sand of diameter d could bring the angle of the sandpile from θ_r to a value $\theta > \theta_m$, the condition being that $L < d/(\theta_m - \theta_r) \approx 30d$.

Contrary to the results of Held *et al.*, Rosendahl, Vekić, and Kelley [11] found a persistent power-law behavior of the small avalanches over the rim in the sandpile, independent of system size: As the system increases in size, large avalanches appear and they eventually dominate the mass flow over the rim, but the power-law distribution of small avalanches persists. However, the data presented in Ref. [11] indicate that the cutoff in the power-law distribution of small avalanches does not scale with system size.

In an experiment of type (a) by Bretz *et al.* [9], the avalanches that occur down the slope are studied. They also find large sliding events occurring regularly, but in addition they find smaller avalanches with a power-law distribution of sizes. However, no finite-size scaling analyses were done, and it would be very interesting to see whether the power-law distribution of avalanche sizes scales with system size, since this is a unique fingerprint of a system displaying SOC.

There is a striking discrepancy between the experiments in Refs. [7,8,10,11] and the sandpile cellular automaton. When measuring the flow of sand over the rim, the internal avalanches are neglected [13,10]. The latter experiment [9] is the only experiment where the avalanches down the slope are studied. This is the relevant measure if we want to make a close connection to the sandpile cellular-automaton model. Also, the experiments of type (a) are essentially one dimensional while the model sandpile is two dimensional. An additional complication, which is not taken into account in the original sandpile cellular-automaton model, is the question of inertia effects. Including the inertia effects Prado and Olami [14] have reproduced the experimental results of Refs. [10,11] for the distribution of flow over the rim.

The study of the SOC systems has to a great extent been based on simulations that use cellular-automata models. The majority of these simulations have been limited to *conservative* models, i.e., models where the simple dynamical rules conserve the dynamical variable. It has been suggested that the necessary (and sufficient) condition for SOC is indeed the conservation law [15,16], but recently a class of nonconservative models was shown to display self-organized criticality as well [17].

Section II is devoted to a discussion of sandpile models in which we neglect the spatial correlations. The models, known as "random-neighbor models," have no underlying spatial structure defining the neighborhood relations. We show that the random-neighbor model organizes into a statistically stationary state where the average rate of flow of sand into the system equals the average rate of flow of sand out of the system. Given the amount of dissipation (for a precise definition see Sec. II) we can easily calculate the average avalanche size. Furthermore, we show that the random-neighbor model is identical to a branching process where an analytical expression for the avalanche size distribution is known. According to this result, only conservative systems are able to exhibit critical behavior. This highlights the importance of correlations for the discovered existence of critical nonconservative models [17].

In Sec. III we define the sandpile models with an underlying lattice, i.e., we introduce spatial correlations. These systems also reach a statistically stationary state where the average rate of flow of sand into the systems equals the average rate of flow of sand out of the systems.

Thus the average avalanche size has to diverge in the thermodynamic limit where the system size goes to infinity. This is compatible with a power-law distribution of avalanche sizes, the existence of which is further supported by additional arguments related to the transport properties of the system.

We discuss the host of critical exponents related to the sandpile model. We review the scaling relations and additional constraints addressed by several authors [18–20]. We find that the proposed values for the critical exponents are not self-consistent [18,19] or, in the case of Ref. [20], not compatible with measurements (the reason being that the value of one of the critical exponents is based on a wrong assertion). We present a coherent set of equations leaving only one critical exponent to be determined. We estimate one exponent from simulations of the model. Determining the other exponents from the self-consistent set of equations we find excellent agreement between the measured and conjectured values.

II. SANDPILE MODELS WITHOUT SPATIAL STRUCTURE

A. Definition of the model

Given N sites numbered $i=1,\ldots,N$, an integer variable z_i is associated with every site i. All sites are capable of storing $z_{\text{th}}-1$ units. The system is *perturbed* by adding one unit at a time to a randomly chosen site, that is, $z_i \to z_i+1$. We stop perturbing the system if, at some point, $z_i \ge z_{\text{th}}$: the site *topples*, i.e., its content is distributed to "neighboring sites" or simply dissipated. To be more precise, if z_i exceeds the threshold value z_{th}, then the site relaxes $z_i \to z_i - z_{\text{th}}$ and we add one unit to αz_{th} randomly chosen neighbors

$$z_{j_k} \to z_{j_k}+1 \ , \quad k=1,\ldots,\alpha z_{\text{th}} \ . \tag{1}$$

The parameter α determines the number of random neighbors, its form being $\alpha = l/z_{\text{th}}$, where $l \in \{0,\ldots,z_{\text{th}}-1\}$.

We can choose new random neighbors every time site i topples, in which case we refer to the model as an *annealed* random-neighbor model. The random choice of neighbors can also remain fixed during the simulations, which is then called a random-neighbor model with *quenched randomness.* In the following we will restrict ourselves to the annealed model because it allows the result to be derived very easily.

The dynamics of the model are defined as follows: We take a random initial configuration $\{z_i\}_{i=1}^N$, $z_i < z_{\text{th}} \forall i$. We add one unit at a time to a randomly chosen site i, i.e., $z_i \to z_i+1$. If $z_i \ge z_{\text{th}}$ the system relaxes according to the rule

$$\begin{aligned} &z_i \to z_i - z_{\text{th}} \ , \\ &z_{j_k} \to z_{j_k}+1 \ , \quad k=1,\ldots,\alpha z_{\text{th}} \ . \end{aligned} \tag{2}$$

This may cause one or more of the neighboring sites to exceed the threshold value, in which case they have to relax simultaneously (i.e., we use a parallel updating of the lattice) according to Eq. (2), and we say that an avalanche propagates in the system. This process continues until we regain a static state $\{z_i\}_{i=1}^N$, $z_i < z_{\text{th}} \forall i$. Then we perturb the system once again and so on.

B. The statistically stationary state

Let P_z be the probability that a given site contains z units. One toppling can cause 0 to αz_{th} new topplings. It will cause a neighbor to topple if the neighbor has $z_{\text{th}}-1$ units. Since the neighbors are randomly chosen, it will cause b topplings with the probability

$$p_b = \begin{bmatrix} \alpha z_{\text{th}} \\ b \end{bmatrix} P_{z_{\text{th}}-1}^b (1-P_{z_{\text{th}}-1})^{\alpha z_{\text{th}}-b} \ , \quad b=0,\ldots,\alpha z_{\text{th}} \tag{3}$$

averaging the number of new topplings to

$$\langle b \rangle = \sum_{b=0}^{\alpha z_{\text{th}}} b p_b = \alpha z_{\text{th}} P_{z_{\text{th}}-1} \ . \tag{4}$$

If $\alpha<1$ and $N \to \infty$ this model does not form loops (the probability that a toppling site is chosen as a neighbor by one of its own neighbors is of the order of $1/N$) and the random-neighbor model is considered a true branching process. The probability of creating b new branches is given by Eq. (3), and on the average $\langle b \rangle = \alpha z_{\text{th}} P_{z_{\text{th}}-1}$ branches are created. Very quickly the system settles into a statistically stationary state in which $\langle z \rangle = (1/N)\sum_{i=1}^N z_i$ fluctuates around a constant value.

Note that when $\alpha=1$ the random-neighbor model is not well defined since $\langle z \rangle$ will continue to grow until $\langle z \rangle = z_{\text{th}}$. The following perturbation will initiate an avalanche which goes on forever. However, the branching process is mathematically well defined.

It is easily shown that the system reaches an equilibrium state, where the rate of flow into a state z equals the rate of flow out of that same state. During an avalanche (or perturbation) the rate of flow into the state z is proportional to the probability that a chosen site contains exactly $z-1$ units, i.e., P_{z-1}, while the rate of flow out of the state z is proportional to P_z (the constants of proportionality are the same, namely αz_{th} times the total number of toppling lattice sites at that moment). If $P_{z-1} > P_z$ ($P_{z-1} < P_z$), the rate of flow into the state z is larger (smaller) than the rate of flow out of the same state, and P_z will increase (decrease) until $P_{z-1} = P_z$. Thus

$$P_{z-1} = P_z \ , \quad z=1,\ldots,z_{\text{th}}-1 \tag{5}$$

is an *attractor* of the system. This is verified by computer simulations of the annealed random-neighbor model.

Using the renormalization condition $\sum_{z=0}^{z_{\text{th}}-1} P_z = 1$, we find that

$$P_z = \frac{1}{z_{\text{th}}} \ , \quad z=0,\ldots,z_{\text{th}}-1 \ . \tag{6}$$

We notice that the organization of the system is independent of α because the derivation of Eq. (6) does not rely on the value of α. This separation of the dynamics of the avalanches from their organization of the medium through which they propagate is probably the reason why it is possible to obtain analytical results. Using Eq. (4) we find that the average number of new topplings is determined by the parameter α

$$\langle b \rangle = \alpha \ . \tag{7}$$

C. The avalanche size distribution

The parameter α determines the amount of nonconservation in the system. If $\alpha < 1$ we dissipate $z_{\rm th} - \alpha z_{\rm th}$ units in every toppling. In a statistically stationary state we must dissipate as many units as we put into the system. Let s denote the size of an avalanche (i.e., the total number of topplings in an avalanche) and let $\langle s \rangle$ be the average size of an avalanche. Then

$$\langle s \rangle (z_{\rm th} - \alpha z_{\rm th}) = \frac{1}{P_{z_{\rm th}-1}} = z_{\rm th} \tag{8}$$

since $1/P_{z_{\rm th}-1}$ is the average rate of flow into the system (the average number of additions before we trigger an avalanche). Thus the average avalanche size is given by

$$\langle s \rangle = \frac{1}{1-\alpha} \ . \tag{9}$$

Using the analogy with branching processes we can express the distribution function of avalanche sizes s analytically (see, e.g., Ref. [21]),

$$P(s) \propto s^{1-\tau_s} \exp\left[-\frac{s}{s_\xi} \right] = s^{-3/2} \exp\left[-\frac{s}{s_\xi(\alpha)} \right] ,$$
$$s_\xi(\alpha) \propto \frac{1}{(1-\alpha)^2} , \tag{10}$$

where we introduce the *power-law exponent* τ_s and the *cutoff in the cluster size distribution* s_ξ. Note that Eq. (10b), along with the left-hand side of Eq. (10a), implies that $\tau_s = \frac{5}{2}$.

When $\alpha = 1$ the model is conservative with $\langle s \rangle = \infty$ and $s_\xi = \infty$, following that $P(s) \propto s^{-3/2}$; the system is critical. In terms of the branching process, the average number of new topplings $\langle b \rangle = 1$, i.e., the (critical) branching process is just barely able to continue. When $\alpha < 1$ the model is nonconservative with $\langle s \rangle = 1/(1-\alpha) < \infty$ and $s_\xi < \infty$, following that $P(s) \propto s^{-3/2} \exp(-s/s_\xi)$; the system is subcritical; see, e.g., Ref. [22].

Figure 1(a) is a graph of the distribution function of avalanche sizes, obtained from a test simulation with $N = 50\,000$ and $z_{\rm th} = 20$ when $\alpha = \frac{2}{20}$ and $\frac{18}{20}$. We have fitted the measured distribution function to the form given by Eq. (10) and plotted the cutoff in cluster size distribution s_ξ against $1/(1-\alpha)^2$ in Fig. 1(b).

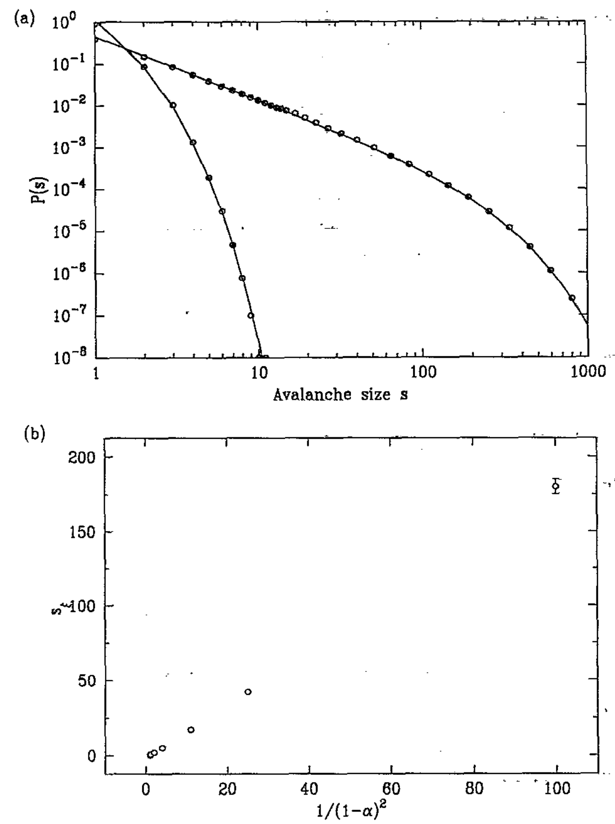

FIG. 1. (a) The measured distribution function of avalanche sizes in the annealed random-neighbor model with 2 and 18 neighbors. The solid lines show the analytical expression of Eq. (10) when we use $s_\xi = 0.64$ (2 neighbors) and 180 (18 neighbors). (b) The cutoff in cluster size distribution s_ξ as a function of $1/(1-\alpha)^2$.

III. SANDPILE MODELS WITH A SPATIAL STRUCTURE

A. Definition of the model

Let the geometry of the model (i.e., the neighborhood relations) be given by an underlying lattice. We assign an integer z_i to each lattice site i, where $i = 1, \ldots, N$. The integer z_i represents an appropriate dynamical variable (e.g., slope, mechanical stress, heat, pressure, or energy) in site i in a spatially extended system. In the following we refer to z_i as the height of a column of sand in site i. We perturb the system (add sand to the system) by choosing at random a position increasing the dynamical variable with one unit, i.e., $z_i \rightarrow z_i + 1$.

Whenever the dynamical variable in site i exceeds a threshold value, the site topples. Let C denote the coordination number of an interior point in the lattice and let i_k, $k = 1, \ldots, C_i$ denote the nearest neighbors of site i with the coordination number C_i. For reasons of simplicity, we choose $z_{\rm th} = C$ in the following. It implies that the allowed values in site i are $0, \ldots, z_{\rm th} - 1$. The process of a toppling in site i is defined by

$$z_i \rightarrow z_i - C \ ,$$
$$z_{i_k} \rightarrow z_{i_k} + 1 \ , \quad k = 1, \ldots, C_i \ . \tag{11}$$

As a result one or more neighbors may exceed the threshold value in which case they have to relax and an avalanche will propagate in the system. This model is defined as *undirected* since the local toppling rule is an isotropic mechanism. We describe the model as being *directed* if the toppling is anisotropic.

The toppling rule conserves the sum of the z values whenever an interior site topples ($C_i = C$). Dissipation only occurs when a boundary site topples ($C_i < C$) because a boundary site has fewer nearest neighbors than a site within the system.

The attractor of the system

Analogous to the behavior of the random-neighbor model, this system will reach an equilibrium state where, on the average, the rate of flow into the system equals the rate of flow out of the system across the boundary. That is, $\langle z \rangle = (1/N)\sum_{i=1}^{N} z_i$ fluctuates around a constant value. The system is very sensitive to perturbations if $\langle z \rangle$ becomes too large. Sand will rapidly be transferred to the boundary, where it dissipates. On the other hand, the system is not as sensitive to perturbations if $\langle z \rangle$ becomes too small. These two situations—to maximize the average z value and to stabilize the system by dissipating sand at the boundary—maintain an equilibrium situation. Pietronero, Tartaglia, and Zhang have shown that it is possible to calculate $\langle z \rangle$ by using an "effective-medium theory," which relies on the above-mentioned balance between the rate of flow into and out of the system [19]. Unfortunately we cannot apply the argument we used in the case of the random-neighbor model which leads to a statistically stationary state characterized by $P_0 = \cdots = P_{z_{\text{th}}-1}$, since the proof relies on an assumption of totally uncorrelated sites.

B. The implications of the spatial structure

The introduction of the spatial structure to the model has two implications. First, the spatial correlations will introduce correlations between the values in the lattice sites. That is, not all of the $(z_{\text{th}})^N$ stable configurations are allowed when the system has reached the equilibrium state [23]. For example, two nearest neighbors cannot both be equal to 0. Suppose $z_i = 0$. The site has just relaxed. This implies that all the nearest neighbors $z_{i_k} \geq 1$, $k = 1, \ldots, C_i$ since one unit is transferred to each of the nearest-neighboring sites by the relaxation rule. Dhar has explicitly calculated the number of states in the attractor $N_{\mathfrak{R}}$ (known as the number of recurrent states) and shown that these states occur with equal probability [23]. The entropy associated with the SOC state is thus $S = \ln N_{\mathfrak{R}}$. With $z_{\text{th}} = 4$ only $(3.210\ldots)^N$ states out of the 4^N stable states are allowed when the system has reached the statistically stationary state [23,24]. Thus the system self-organizes into an exponentially small subset, which is to be identified with the attractor of the dynamics. (In the random-neighbor model all stable states are allowed, i.e., $N_{\mathfrak{R}} = 4^N$.)

It is possible to get more quantitative results. Majumdar and Dhar have analytically calculated the probability that two sites separated by a distance r in a d-dimensional hypercubic lattice would both have the minimum value (0 in our definition) [25]. They find an anticorrelation which, to the lowest order in r, varies as r^{-2d} for large r. In the case of the Bethe lattice the anticorrelations decrease exponentially as Dhar and Majumdar have shown in Ref. [26]. In truly infinite dimensions (the branching process) these correlations vanish completely, by definition.

Second, the question of dissipation is transformed to a transport problem where the input has to be transported from neighbor to neighbor until it reaches a site where it dissipates—in this case, at the boundary. If no dissipation takes place at all (e.g., if the model is defined with periodic boundary conditions), the model is not well defined as was the case with the random-neighbor model, i.e., eventually an initiated avalanche will go on forever.

C. Scaling arguments

1. Scaling of the average avalanche size

In the following we restrict the discussion to d-dimensional hypercubic lattices of linear size L ($N = L^d$). The system can only dissipate energy through the boundary. The average distance a particle has to travel to reach the boundary is proportional to L when we deposit the units at random. This implies that

$$\langle s \rangle \geq K(L)L \ , \tag{12}$$

where $K(L)$ is an L-dependent function. Because of the random deposition we must have $\langle s \rangle \propto L^{\epsilon}$, $\epsilon \geq 1$, i.e., $dK/dL \geq 0$. The specific form of the function K depends on the actual relaxation dynamics. Thus the average size of avalanches scales with system size and is infinite in the thermodynamic limit. If the particle diffuses out to the boundary, the added particle has to perform L^2 steps to cover the distance L as pointed out by Kadanoff *et al.* [27], i.e., $K(L) = BL$, where B is a constant:

$$\langle s \rangle \geq BL^2 \ . \tag{13}$$

This is indeed the case for the BTW sandpile relaxation given by Eq. (11). Dhar has given an analytical proof that $\langle s \rangle \propto L^2$ in the undirected BTW-type relaxations independent of dimension [23]. This has been verified by measurements of Grassberger and Manna [28].

In a directed model $\langle s \rangle \propto L$ so $K(L) = \tilde{B}$, see, for example, Ref. [27]. This can be shown analytically by copying the proof of Dhar given for the undirected version of the model.

Let $P(s)$ denote the distribution function of avalanche sizes. By definition

$$\langle s \rangle = \int sP(s)ds \ , \tag{14}$$

following the fact that $P(s)$ must be of a form producing an infinite average value in the thermodynamic limit $L \to \infty$. Motivated from the discussion of branching processes (and from percolation theory among others) we postulate a distribution function of the form

$$P(s) \propto s^{1-\tau_s} \exp\left[-\frac{s}{s_\xi(L)}\right] . \tag{15}$$

The distribution function cannot decrease exponentially in the thermodynamic limit since this will make the average avalanche size finite. A pure power-law behavior, however, produces a divergence of the average avalanche size (provided the power-law exponent τ_s is smaller than 3). Thus the cutoff in cluster size distribution has to go to infinity when $L \rightarrow \infty$ to ensure that the average value becomes infinite:

$$s_\xi(L) \rightarrow \infty \quad \text{when } L \rightarrow \infty \ . \tag{16}$$

Only in exceptional cases is it possible to prove that the distribution function is a power law in the thermodynamic limit: the Bethe lattice where $\tau_s = \frac{5}{2}$ [26] and the directed model with $\tau_s = \frac{7}{3}$ in $d=2$ and $\tau_s = \frac{5}{2}$ in $d \geq 3$ [29]. In all other models we have to rely on simulations in order to convince possible skeptics of the power-law behavior of the distribution function of avalanche sizes.

Because we are unable to prove the existence of power-law behavior we are also unable to get analytical expressions for the power-law exponent. However, if we assume a power-law behavior we can obtain some limits for the power-law exponent τ_s as well as relate the critical exponent to other critical exponents yet to be defined.

2. *Finite-size scaling and scaling relations*

Let $P(s,L)$ denote the distribution function in a system with linear size L. Suppose that the distribution function is a power law up to a certain cutoff size which depends on the system size L. If this dependence is of a power-law type, then a generalization of Eq. (15) is the *finite-size scaling ansatz*

$$P(s,L) \propto L^{-\beta} g\left[\frac{s}{L^\nu}\right] , \tag{17}$$

where g is a universal scaling function and β and ν are critical indices. ν describes how the cutoff size scales with system size. By rewriting the finite-size scaling ansatz

$$\begin{aligned} P(s,L) &\propto L^{-\beta}\left[\frac{s}{L^\nu}\right]^{-\beta/\nu} \tilde{g}\left[\frac{s}{L^\nu}\right] \\ &\propto s^{-\beta/\nu}\tilde{g}\left[\frac{s}{L^\nu}\right] \end{aligned} \tag{18}$$

we easily see that it is a generalization of the former ansatz in Eq. (15). As a by-product, we get a *scaling relation*, i.e., a relation between the three critical exponents

$$1-\tau_s = -\frac{\beta}{\nu} \ . \tag{19}$$

We can take advantage of the knowledge that for undirected model $\langle s \rangle \propto L^2$, while for directed models $\langle s \rangle \propto L$—valid in all dimensions—to get an additional scaling relation

$$\begin{aligned} \langle s \rangle &= \int_1^\infty sP(s,L)ds \Big/ \int_1^\infty P(s,L)ds \\ &= \int_1^\infty sL^{-\beta} g\left[\frac{s}{L^\nu}\right] ds \\ &= L^{2\nu-\beta}\int_{1/L^\nu}^\infty \tilde{s}g(\tilde{s})d\tilde{s} \\ &\rightarrow L^{2\nu-\beta} \quad \text{for } L \rightarrow \infty \ . \end{aligned} \tag{20}$$

To perform the last step we assume that the integral converges. Therefore we have

$$2\nu-\beta = \begin{cases} 2 & (\text{undirected}) \\ 1 & (\text{directed}) \ . \end{cases} \tag{21}$$

Eliminating β using Eq. (19) we get

$$\nu(3-\tau_s) = \begin{cases} 2 & (\text{undirected}) \\ 1 & (\text{directed}) \ , \end{cases} \tag{22}$$

which once again shows that $\tau_s \leq 3$ since $\nu \geq 0$.

We can induce further scaling relations between additional critical exponents. Let s, t, r, and a denote the variables corresponding to the size, lifetime, radius (linear size), and area of the avalanches, respectively, We assume a power-law behavior of the probability densities, that is,

$$P(x) \propto x^{1-\tau_x} , \tag{23}$$

where $x \in \{s,t,r,a\}$. The existence of a relation between the different variables implies the existence of scaling relations between the exponents τ_s, τ_t, τ_r, and τ_a. Let γ_{xy} denote the exponent relating x with y, i.e.,

$$x \propto y^{\gamma_{xy}} \ . \tag{24}$$

Thus

$$\begin{aligned} P(x)dx &= P(y)dy \\ \Longrightarrow x^{1-\tau_x} &\propto y^{1-\tau_y}\frac{dy}{dx} \\ \Longrightarrow x^{1-\tau_x} &\propto x^{(1-\tau_y)/\gamma_{xy}} x^{(1/\gamma_{xy})-1} , \end{aligned} \tag{25}$$

resulting in

$$\tau_x = 2 + \frac{\tau_y - 2}{\gamma_{xy}} \ . \tag{26}$$

According to Eq. (24) $y \propto x^{1/\gamma_{xy}}$ so by definition

$$\gamma_{xy}^{-1} = \gamma_{yx} \ . \tag{27}$$

Also, we find that

$$x \propto y^{\gamma_{xy}} \propto (z^{\gamma_{yz}})^{\gamma_{xy}} \Longleftrightarrow \gamma_{xy}\gamma_{yz} = \gamma_{xz} \ . \tag{28}$$

The exponents γ_{xy} can be measured as well. They appear as the exponents of the *conditional expectation values* [30]. For example,

$$E[s|t] \underset{\text{def}}{=} \sum_s sP(s|t) \propto t^{\gamma_{st}} , \tag{29}$$

where $P(s|t)$ is the *conditional density* of s given t. The reciprocal relationship between the exponents relating the different variables is a necessary condition if we assume the existence of a transformation between the variables. In a strictly mathematical sense such transformations cannot exist since, as an example, there will always be more than one possible avalanche size s for a given lifetime t. However, if we find that the reciprocal relationships are fulfilled quite accurately, it indicates that the conditional densities, e.g., $P(s|t)$, have a narrow support around their average value.

We have 16 unknown exponents altogether, namely τ_x and γ_{xy} where $x,y\in\{s,t,r,a\}$ with $x\neq y$, but there exists only 12 linearly independent equations in the form of Eqs. (26) and (27). Equation (28) does not contain any new information; however, we can add two additional equations.

3. *Scaling exponent determined by compactness and isotropy*

In the remaining part, we restrict the discussion of the two-dimensional BTW model. It is quite easy to prove that the avalanches are compact. No holes are allowed inside an avalanche if the system has reached the critical state. We apply the concept of *forbidden subconfiguration* introduced by Dhar to prove the compactness [23]. Let $z_{\rm th}=C$, where C is the coordination number of an interior lattice site. Any set $\mathbb{F}$ with heights satisfying

$$z_i\leq C_i-1\ ,\quad \forall z_i\in\mathbb{F} \tag{30}$$

is called a forbidden subconfiguration. A configuration $\{z_i\}$ is an *allowed subconfiguration* only if it does not contain any forbidden subconfigurations. A system is in the critical state if and only if it is an allowed configuration [23].

The characterization of an allowed state enables us to define an algorithm—known as the *burning algorithm*—to determine whether a given state is critical. Let T be a "test" state. T is a forbidden state if all sites i in T satisfy $z_i\leq C_i-1$. Otherwise, there are some sites j in which $z_j>C_j-1$. Let $T^{(1)}$ denote the smaller set we obtain when deleting (burning) all these sites. If possible we repeat the burning process in $T^{(1)}$, but at some point we cannot continue the burning process: The final set $T^{(n)}$ is either an empty set, in which case T is a critical state, or a nonempty set, in which case T is not a critical state.

First, we prove that avalanches are compact in a two-dimensional square lattice with $z_{\rm th}=4$. Suppose that we do have an avalanche with holes inside. Let $\mathcal{S}$ be a connected part without any topplings inside the avalanche and let $\tilde{\mathcal{S}}$ denote the same set just before the avalanche. If $N_{\rm EN}$ denotes the number of exterior neighbors then the boundary sites of $\tilde{\mathcal{S}}$ satisfy the inequality

$$\tilde{z}_b\leq 3-N_{\rm EN} \tag{31}$$

since they do not topple; see Fig. 2.

Let C_i denote the coordination number of sites within $\tilde{\mathcal{S}}$. Using

$$C_i+N_{\rm EN}=4\ , \tag{32}$$

the boundary sites of $\tilde{\mathcal{S}}$ satisfy

$$\tilde{z}_b\leq C_i-1\ . \tag{33}$$

The sites $\tilde{z}$ in the interior of $\tilde{\mathcal{S}}$ also fulfill the condition $\tilde{z}\leq C_i-1$, so $\tilde{\mathcal{S}}$ is a forbidden subconfiguration. Thus the system was not in the critical state when the avalanche was initiated.

We can prove the compactness of avalanches in an arbitrary lattice by use of the principle of induction: If $\mathcal{S}$ consists of just one site $\tilde{\mathcal{S}}$ is not an allowed subconfiguration since

$$\tilde{z}\leq z_{\rm th}-1-N_{\rm EN}=-1\ . \tag{34}$$

Assume that if $\mathcal{S}$ consists of M sites, then $\tilde{\mathcal{S}}$ is not an allowed subconfiguration. Then $\mathcal{S}$ cannot consist of $M+1$ sites either, since $\tilde{\mathcal{S}}$ will not be an allowed subconfiguration. Adding one site to a set which is not allowed beforehand does not make it an allowed set. Using the compactness along with the isotropic nature of the avalanches we conclude that

$$\gamma_{ar}=2\ , \tag{35}$$

but we are still short of three equations in order to determine all the critical exponents.

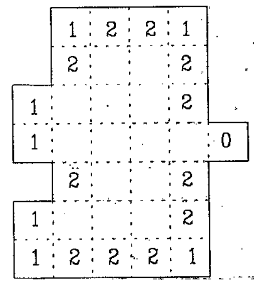

FIG. 2. To prevent topplings inside $\mathcal{S}$ the boundary sites of $\tilde{\mathcal{S}}$ cannot be larger than 3 minus the number of exterior neighbors. The figure displays a situation in which the boundary sites equal the maximum value. By inspection we see that the configuration $\tilde{\mathcal{S}}$ is a forbidden subconfiguration, i.e., $\tilde{\mathcal{S}}$ is not burnable.

4. *Scaling relation induced by central site multiple topplings*

Majumdar and Dhar supply us with an additional equation [20]. Using the analytical result that the average number of topplings at the site initiating the avalanche $\langle n_c\rangle$ scales with the linear system size L as

$$\langle n_c\rangle\propto\log_{10}(L)\ , \tag{36}$$

and assuming that

$$s\propto an_c\ , \tag{37}$$

it is possible to show that

$$\gamma_{sr}=\tau_r\ . \tag{38}$$

Substituting Eq. (38) into Eq. (26) with $(x,y)=(s,r)$ we find that

$$\tau_s=2+\frac{\tau_r-2}{\tau_r}\Longleftrightarrow 2=\tau_r(3-\tau_s)\ , \tag{39}$$

showing that Eq. (38) is equivalent with $\nu=\tau_r$; see Eq. (22). The relation $s\propto r^{\gamma_{sr}}=r^{\tau_r}$ shows the origin of $\tau_r=\nu$ since the cutoff in avalanche size scales with system size as L^{ν}. Thus we are "only" short of two equations.

TABLE I. The critical exponents defined in Eqs. (23) and (24) in the undirected Abelian sandpile model. The first column is the result of Majumdar and Dhar based on the assertion that $\tau_a=\frac{13}{6}$ and $\gamma_{tr}=\frac{5}{4}$ [20]. The second column shows the values estimated from the scaling relations along with the percolation results in 2D, i.e., $\tau_s=\frac{187}{91}$, $\gamma_{sr}=\frac{91}{48}$, and $\gamma_{sa}=1$. The third column is the exponents given by Zhang [18] and Pietronero, Tartaglia, and Zhang [19]. From measurements of τ_s they estimate that $D\approx 1.17$. The estimates in the fourth column rely on the ansatz $\tau_t=\frac{15}{7}$ and $\gamma_{tr}=\frac{4}{3}$ where the latter is based on the argument by Zhang [18]. Finally, we list the measured exponents in a $L=100$ lattice; see Fig. 4.

Exponent	Ref. [20]	Percolation	Refs. [18,19]		Measured ± 0.05
τ_s	$\frac{15}{7}\approx 2.14$	$\frac{187}{91}\approx 2.05$	$2+\dfrac{D-2/3}{D+4/3}$	$\frac{48}{23}\approx 2.09$	2.08
τ_t	$\frac{34}{15}\approx 2.27$			$\frac{15}{7}\approx 2.14$	2.14
$\tau_r=\nu=1+D$	$\frac{7}{3}\approx 2.33$	$\frac{101}{48}\approx 2.10$	2	$\frac{46}{21}\approx 2.19$	2.14
τ_a	$\frac{13}{6}\approx 2.17$	$\frac{187}{91}\approx 2.05$	2	$\frac{44}{21}\approx 2.10$	2.07
γ_{st}	$\frac{28}{15}\approx 1.87$			$\frac{23}{14}\approx 1.64$	1.63
γ_{sr}	$\frac{7}{3}\approx 2.33$	$\frac{91}{48}\approx 1.90$	$D+\frac{4}{3}$	$\frac{46}{21}\approx 2.19$	2.20
γ_{sa}	$\frac{7}{6}\approx 1.17$	1		$\frac{23}{21}\approx 1.10$	1.06
γ_{tr}	$\frac{5}{4}=1.25$		$\frac{4}{3}$	$\frac{4}{3}\approx 1.33$	1.36
γ_{ta}	$\frac{15}{24}\approx 0.63$			$\frac{2}{3}\approx 0.67$	0.67
γ_{ra}	$\frac{1}{2}=0.50$	$\frac{48}{91}\approx 0.53$	$\frac{1}{2}$	$\frac{1}{2}=0.50$	0.50

5. *The dynamical exponent γ_{tr}*

A possible way out of the shortfall of equations is either to establish a connection between the BTW model and other models in which some of the exponents are known analytically or to use additional information about the model (for example, the transport properties) to derive further constraints on the critical indices.

Majumdar and Dhar have shown that each allowed configuration in the sandpile model corresponds to a spanning tree [20]. Taking advantage of the known relationship between the latter and the q-state Potts model they establish an equivalence between the sandpile model and the $q\to 0$ limit of the q-state Potts model.

The dynamical exponent γ_{tr} in a spanning tree equals $\frac{5}{4}$. Majumdar and Dhar use this value as the value of the dynamical exponent in the BTW model. Thus they only have to measure one exponent to determine all the others. They use the ansatz $\tau_a=\frac{13}{6}$ in order to calculate the remaining critical exponents. We list the findings in Table I.

However, the dynamical exponent in the spanning tree reflects how a spanning tree burns using the burning algorithm. Thus the dynamical exponent $\gamma_{tr}=\frac{5}{4}$ is related to a (static) geometrical description of a spanning tree, but has nothing to do with the dynamics of getting from one critical state of the sandpile to another.

Zhang has proposed $\gamma_{tr}=\frac{4}{3}$ based on heuristic arguments [18]. The idea is to view the avalanche as a collection of *activation fronts* that propagates in the lattice. The activation front at time τ is defined as the toppling sites at time τ. If the activation front was to perform a pure random walk

$$t\propto r^{\gamma_{tr}}=r^2\ . \tag{40}$$

However, sites which have just toppled tend to repel the activation fronts. Thus the avalanche prefers to expand outward and we expect $\gamma_{tr}<2$. If the activation front performs a self-avoiding random walk it implies

$$\gamma_{tr}=\tfrac{4}{3}\ . \tag{41}$$

6. *The interpretation of the fractal boundary*

Even though the avalanches are compact, it turns out that the boundaries are fractal, i.e., the length of the boundaries scales with the radius of the avalanches like r^D, $D>1$ [28,19]. The following argument shows that the critical exponent τ_r is related to the fractal properties of the boundary of an avalanche. We do not determine

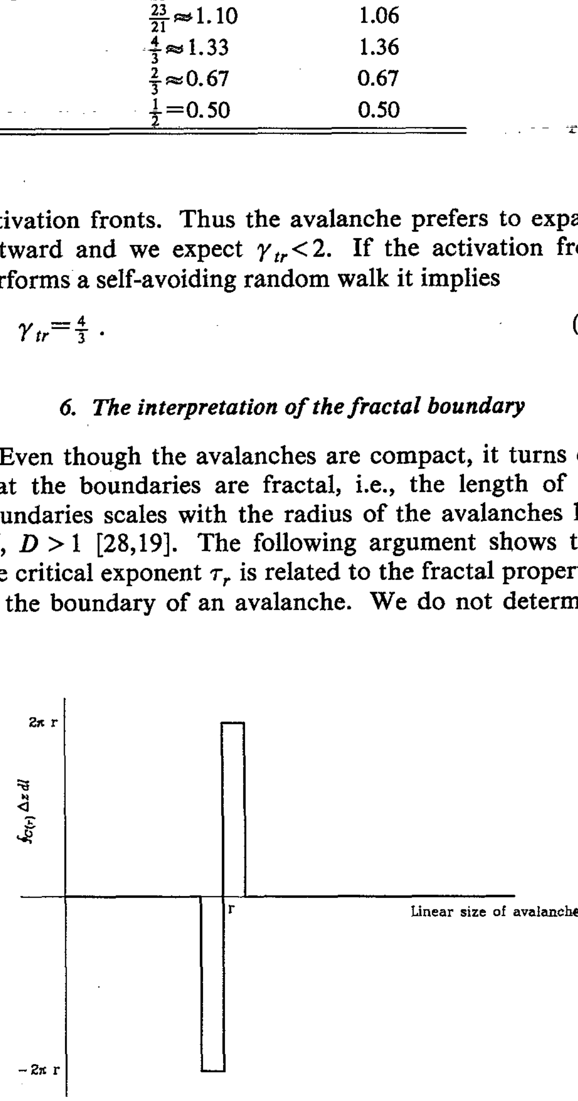

FIG. 3. During an avalanche where the sites topple exactly once only the sites at the boundary change z value. The graph represents the integrated amount of change in z value in an avalanche of linear size r, $\oint_{C(r)}\Delta z\, dl$, where $C(r)$ is the circumference of radius r.

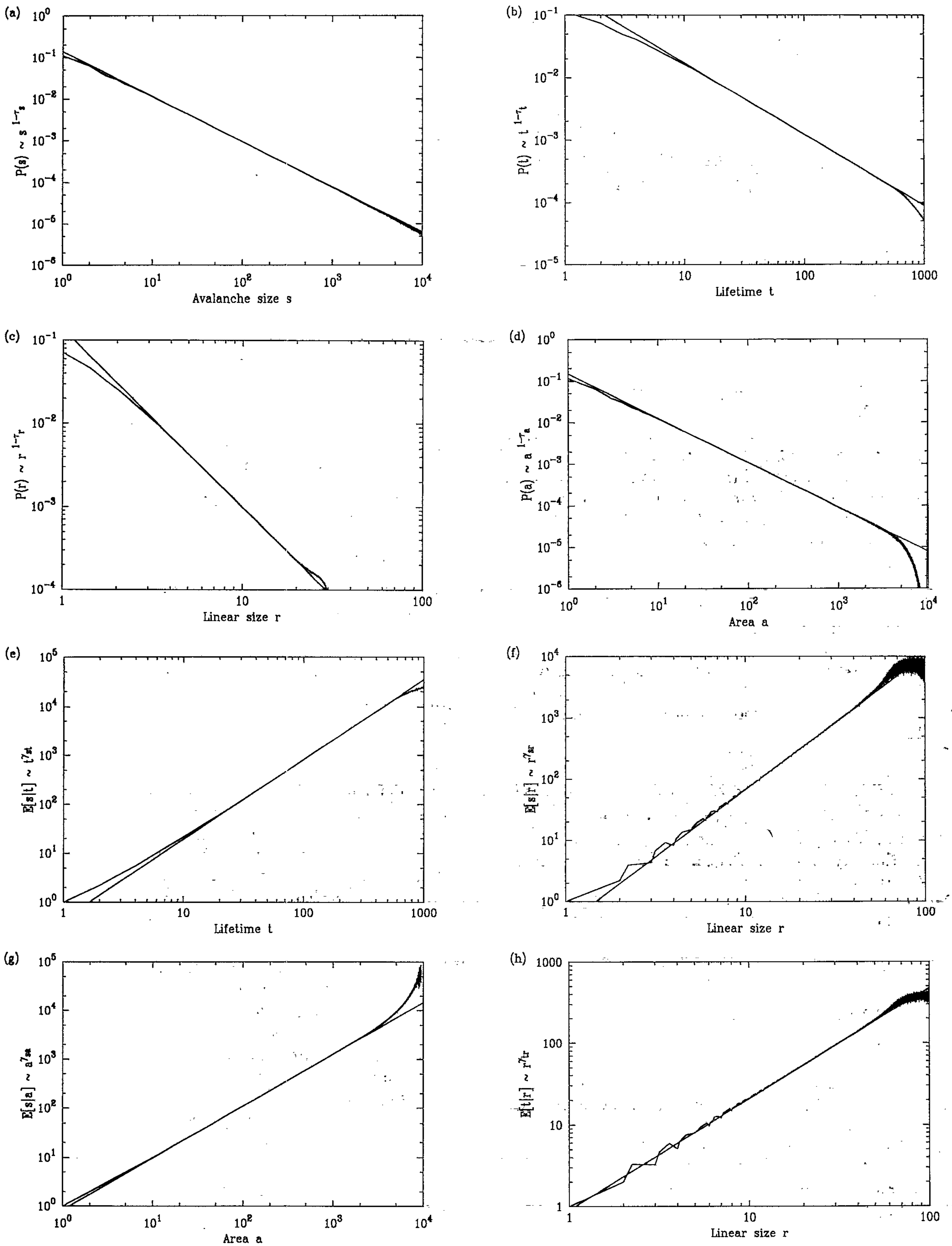

FIG. 4. Simulation results in a two-dimensional system with BTW relaxation rules. The slope of the straight line in each $\log_{10}$-$\log_{10}$ plot determines the critical exponent. All the results are listed in Table I.

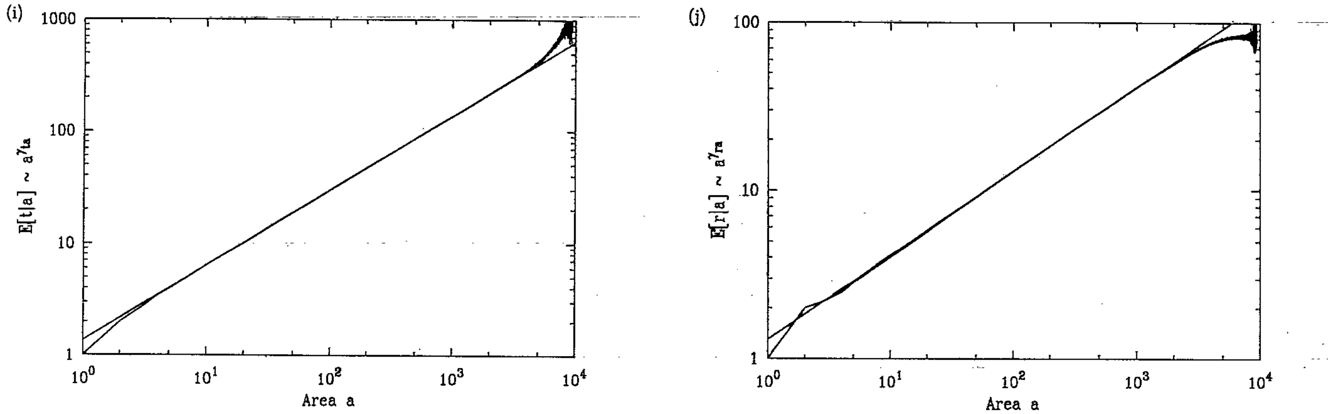

FIG. 4. (*Continued*).

the exponent τ_r but the argument explains why it is larger than 2.

Assume that we perturb a critical system in the center of a fixed circle of radius r. On the average, the rate of flow through the surface of the circle must be a constant. For simplicity we make the additional assumption that the avalanche is circular. It is clear that only avalanches with a radius larger than or equal to r can induce a transport through the surface of the circle.

Let us assume we have a situation where the sites only topple once during an avalanche, i.e., the area of the avalanche is identical to the size of the avalanche ($a=s$). The only transport (change of z values) in such an avalanche is at the avalanche surface. Schematically, the situation is shown in Fig. 3 where it is disclosed that the amount of transport is proportional to the length of the boundary.

Thus

$$P(r)r \propto 1 \Longrightarrow P(r) \propto r^{-1} = r^{1-\tau_r} , \tag{42}$$

in accordance with the suggestion of Zhang [18] and Pietronero, Tartaglia, and Zhang [19]. This argument may, by the way, indicate why power-law behavior is observed in the sandpile models.

However, if the avalanches are compact [Eq. (35)] but the boundary is a fractal, i.e., the length of the boundary is r^D, then

$$P(r)r^D \propto 1 \Longrightarrow P(r) \propto r^{-D} = r^{1-(1+D)} . \tag{43}$$

Thus we identify

$$1+D=\tau_r , \tag{44}$$

that is, the exponent τ_r is intimately linked with the fractality of the boundary. We thus expect that $\tau_r=\gamma_{sr}=\nu>2$.

Grassberger and Manna measured the fractal dimension of the boundary $D=1.21$ [28]. This value agrees with our measurements of the exponent τ_r.

Note that using the result from Eq. (26) we get

$$\tau_a = 2+\frac{D-1}{\gamma_{ar}} . \tag{45}$$

Since $1 \le D \le 2$ we have the inequality

$$2 \le \tau_a \le \tfrac{5}{2} . \tag{46}$$

One could say that the case $D=2$ resembles the random-neighbor model (where $a=s$) in the sense that the avalanche is all boundary, but of course it makes no sense to talk about the radius of an avalanche in the random-neighbor model.

7. *Conjecture of the values of the critical exponents*

In Table I we have listed the values one would obtain with the analytical result for percolation clusters in a two-dimensional (2D) lattice at the percolation threshold. The clusters in a 2D lattice at the percolation threshold are fractal with an exponent $\frac{91}{48}$, which is to be identified with the exponent γ_{sr}. The exponent τ_s describing the percolation cluster size distribution equals $\frac{187}{91}$ in two dimensions, and since multiple topplings never occur (by definition) $s=a$. The exponents with a reference to time t are not defined.

Also, we display the exponents suggested by Zhang [18] and Pietronero, Tartaglia, and Zhang [19] (using our notation [31]). Giving arguments similar to those leading to Eq. (42), they propose $\tau_r=2$, which together with the compactness implies $\tau_a=2$. Furthermore, they argue that $s \propto r^D t \propto r^{D+\gamma_{tr}}$, resulting in $\tau_s=2+(D+\gamma_{tr}-2)/(D+\gamma_{tr})$. The latter is consistent with the scaling relations Eqs. (26) and (38) if we identify $\tau_r=\gamma_{sr}=D+\gamma_{tr}$. However, this implies $D=\tau_r-\gamma_{tr}=2-\frac{4}{3}<1$, which of course is not the case. They did not take into account the fact that the fractal boundary affects the estimate of the exponents τ_r, and moreover, the assertion $s \propto r^D t$ is debatable and certainly not supported by measurements (e.g., it implies $\gamma_{sr}=D+\gamma_{tr}$).

A consistent set of values can be obtained using $\gamma_{ra}=\frac{1}{2}$ (compactness and isotropy) and $\gamma_{tr}=\frac{4}{3}$ (self-avoiding random walk of activation fronts). Estimating $\tau_t=\frac{15}{7}$, we can calculate all the other critical indices via the scaling relations Eqs. (26) and (27). The result is displayed in Table I, in which we also list the measured values. The data were averaged over 10^8 avalanches in a $L=200$ system in the measurements of τ_s and τ_t. The other exponents are measured in a $L=100$ system. Figure 4

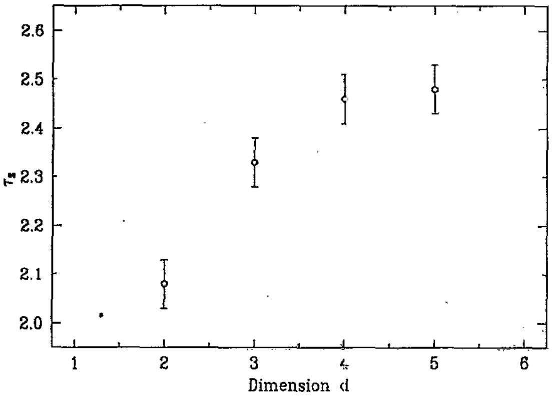

FIG. 5. The power-law exponent τ_s in the BTW sandpile model as a function of dimension d [32]. The exponent approaches the mean-field value of $\frac{5}{2}$ when the dimension increases. We conjecture that $\tau_s=\frac{5}{2}$ when $d \geq d_u=6$.

shows all the measurements listed in Table I. The agreement with the conjectured values is excellent.

D. The upper critical dimension

One would expect the mean-field description of Sec. II to be exact either above an upper critical dimension d_u or in the limit $d \rightarrow \infty$. Simulation of the sandpile model shows that with increasing dimension d the power-law exponent increases towards the mean-field value $\tau_s=\frac{5}{2}$; see Fig. 5 [32]. When $d=5$, $\tau_s=2.48(5)$ (see also Ref. [28]), but it is possible to argue that the upper critical dimension should be larger than or equal to 6 [33]. Thus we conjecture that $d_u=6$ for the BTW sandpile model.

IV. SUMMARY

Models without any spatial correlations are described as branching processes equivalent to the mean-field description. We introduced the random-neighbor models in order to be able to simulate the mean-field theory. These models evolve naturally—self-organize—into a statistically stationary state where the distribution of avalanche sizes is given by

$$P(s) \propto s^{1-\tau_s} \exp\left[-\frac{s}{s_\xi(\alpha)}\right],$$
$$s_\xi(\alpha) \rightarrow \infty \quad \text{for } \alpha \rightarrow 1 . \tag{47}$$

The parameter α can be interpreted as being related to the dissipation in the model. If $\alpha=1$, the model is conservative and the system is critical since the distribution function in Eq. (47) shows a pure power law. Any degree of nonconservation ($\alpha<1$) will introduce a finite cutoff in the cluster size distribution.

The introduction of the neighborhood relations given by an underlying lattice will induce spatial correlations as well as turn the problem into a transport problem. In a Bethe lattice the BTW relaxation rules given by Eq. (11) introduce exponentially decreasing correlations while a hypercubic lattice gives rise to power-law decreasing correlations.

Furthermore, the average size of avalanches will scale with system size. This is a consequence of the transport problem where the rate of flow into the system has to be transported across the boundary, which is the only place dissipation takes place. We observe power-law behavior of the distribution function of avalanche sizes. Heuristic arguments related to the transport properties of the model are given to indicate the origin of the power-law behavior.

Finally, we have derived scaling relations between the host of critical exponents. Additional constraints—(a) the compactness along with the isotropy of avalanches, (b) a relation between the avalanche size and the number of multiple topplings at the origin of the avalanche, and (c) the self-avoiding random walk performed by the activation fronts—are derived. All in all, we have $M-1$ equations to determine the M exponents ($M=16$). Based on simulations we fix one exponent and calculate the remaining exponents according to the relations derived. Thus we conjecture that the power-law exponent corresponding to the distribution of avalanche sizes equals $\frac{48}{23}\approx 2.09$, which is to be compared with our measurements of 2.08(5).

The ultimate goal is (was) of course to produce a closed set of scaling relations and constraints between the critical exponents. One might speculate whether it is possible to exploit the one-to-one correspondence between the SOC states and the spanning trees proved by Majumdar and Dhar [20] to extract the value of one of the critical exponents. This would make the two-dimensional undirected BTW model belong to the class of solvable models which consists of the random-neighbor model, the BTW model on a Bethe lattice, and the directed BTW model.

ACKNOWLEDGMENTS

We would like to thank Per Bak and Henrik Flyvbjerg for enlightening discussions, and we must express our gratitude to Mette Jørgensen and Paul Meakin for a careful reading of the manuscript. K.C. gratefully acknowledge the financial support of the Danish National Science Research Council through Grant No. NATO 11-0174-1. K.C. and Z.O. greatly appreciate the hospitality of Brookhaven National Laboratory and the University of Oslo where this work was accomplished. This work was supported by the Division of Material Science U.S. DOE under Contract No. DE-AC02-76CH00016.

[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. A **38**, 364 (1988).
[2] B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982).
[3] J. Feder, *Fractals* (Plenum, New York, 1988).
[4] P. W. Anderson, Phys. Today **44** (7), 9 (1991).
[5] T. Vicsek, *Fractal Growth Phenomena* (World Scientific, Singapore, 1989).

[6] L. P. Kadanoff, Phys. Today **39** (2), 6 (1986); **44** (3), 9 (1991).
[7] H. M. Jaeger, C.-H. Liu, and S. R. Nagel, Phys. Rev. Lett. **62**, 40 (1989).
[8] S. R. Nagel, Rev. Mod. Phys. **64**, 321 (1992).
[9] M. Bretz, J. B. Cunningham, P. L. Kurczynski, and F. Nori, Phys. Rev. Lett. **69**, 2431 (1992).
[10] G. A. Held, D. H. Solina II, D. T. Keane, W. J. Haag, P. M. Horn, and G. Grinstein, Phys. Rev. Lett. **65**, 1120 (1990).
[11] J. Rosendahl, M. Vekić, and J. Kelley, Phys. Rev. E **47**, 1401 (1993).
[12] C.-H. Liu, H. M. Jaeger, and S. R. Nagel, Phys. Rev. A **43**, 7091 (1991).
[13] H. J. Jensen, K. Christensen, and H. C. Fogedby, Phys. Rev. B **40**, 7425R (1989).
[14] C. P. C. Prado and Z. Olami, Phys. Rev. A **45**, 665 (1992).
[15] T. Hwa and M. Kardar, Phys. Rev. Lett. **62**, 1813 (1989).
[16] G. Grinstein, D.-H. Lee, and S. Sachdev, Phys. Rev. Lett. **64**, 1927 (1990).
[17] Z. Olami, H. J. S. Feder, and K. Christensen, Phys. Rev. Lett. **68**, 1244 (1992).
[18] Y.-C. Zhang, Phys. Rev. Lett. **63**, 470 (1989).
[19] L. Pietronero, P. Tartaglia, and Y.-C. Zhang, Physica A **173**, 22 (1991).
[20] S. N. Majumdar and D. Dhar, Physica A **185**, 129 (1992).
[21] T. E. Harris, *The Theory of Branching Processes* (Springer-Verlag, Berlin, 1963).
[22] P. Bak and H. Flyvbjerg, Phys. Rev. A **45**, 2192 (1992).
[23] D. Dhar, Phys. Rev. Lett. **64**, 1613 (1990).
[24] M. Creutz, Comput. Phys. **5**, 198 (1991).
[25] S. N. Majumdar and D. Dhar, J. Phys. A **24**, L357 (1991).
[26] D. Dhar and S. N. Majumdar, J. Phys. A **23**, 4333 (1990).
[27] L. P. Kadanoff, S. R. Nagel, L. Wu, and S. M. Zhou, Phys. Rev. A **39**, 6524 (1989).
[28] P. Grassberger and S. S. Manna, J. Phys. (Paris) **51**, 1077 (1990).
[29] D. Dhar and R. Ramaswamy, Phys. Rev. Lett. **63**, 1659 (1989).
[30] K. Christensen, H. C. Fogedby, and H. J. Jensen, J. Stat. Phys. **63**, 653 (1991); K. Christensen, in *Spontaneous Formation of Space-Time Structures and Criticality*, edited by T. Riste and D. Sherrington (Kluwer Academic, Dordrecht, 1991), p. 33.
[31] The correspondence between the notation in Ref. [19] and our notation is $\tau = 1-\tau_a$, $\tau' = 1-\tau_s$, $\gamma = D$, and $z = \gamma_{tr}$.
[32] K. Christensen, Ph.D. thesis, University of Aarhus, Denmark, 1992.
[33] K. Christensen, H. Flyvbjerg, and Z. Olami, Phys. Lett. **71**, 2737 (1993).