BTW MODEL IN $d = 2$

The 1987 paper by Bak, Tang and Wiesenfeld [2] introducing the BTW model is one of the most widely cited physics papers in the last twenty years. Since then, various other models inspired by the self-organised criticality paradigm have been introduced [6, 9, 4], connections to absorbing-state phase transitions have been explored [7, 3], and both experimental [8] and real-world data [1, 10] have been scrutinised.

Consider a regular square lattice of size $L \times L$ with a number of integer grain units z_i at each site. The system is driven by adding a unit to a randomly chosen site, $z_i \to z_i + 1$. Whenever z_i exceeds a threshold value, z^{th}, the site topples, $z_i \to z_i - 4$, and each of the four nearest neighbours receives a unit, $z_{\text{nn}} \to z_{\text{nn}} + 1$. An avalanche will propagate until there are no longer any toppling sites, after which the system is driven once again with the addition of a single unit. In the animation $z^{\text{th}} = 4$, and the number of units at a particular site is indicated in greyscale: black for $z_i = 0$, white for $z_i = 3$, and greys for intermediate values. A toppling site is coloured red. A typical way of initialising the system is to populate all sites randomly with a value of z_i between 0 and 3 inclusive. By default, the animation will perform the above rules repeatedly. If you prefer, you may drive the system yourself by choosing the 'sprinkle' option in the 'mode' options. Clicking on a site then adds a single unit (which will initiate an avalanche if you pick a site with $z_i = 3$).

Initially, the average number of grains in the system $\langle z \rangle$, indicated on the left in the animation, increases from 1.5 since the system is able to accommodate more grains: the sandpile is building up. During this transient phase avalanches tend to be small in size. The avalanche size, expressed in the number of toppling sites, is indicated on the left, together with the largest recorded so far. Eventually, however, the system will enter a set of recurrent states, roughly when the amount of grains added to the system equals the amount of grains lost by the system to the edges. In this statistically stationary state the distribution of avalanche sizes is approximately power law, with the largest avalanches being cut-off by the system size. There is, in fact, a remarkable way of determining whether or not a particular configuration of z_i belongs to the set of recurrent states. In the so-called burning algorithm [?], a configuration under consideration has a single unit added to each of its border sites, and two units to each of its corner sites. The system is then allowed to relax under the usual toppling rules until all avalanches have terminated. If each site in the system topples once and once only, then the tested configuration belongs to the set of recurrent states! In the animation, the burning algorithm may be applied by pressing the 'burn' button. Sites that have toppled once and once only are coloured yellow. To restore the configuration, untoggle the 'burn' button. As the system evolves through the transient phase, you can confirm that the area of yellow burnt sites increases. Indeed, by sprinking some grains by hand, you can even try and accelerate the transient phase by targeting patches that remain unburnt.

The standard rules of the BTW model may be slighlty adapted to create rather beautiful and mesmerising patterns. Suppose, instead of driving the system by adding a unit at a random site, we always add at the central site (which explains the unusual choice of system size $L = 149!$). Also, let us initialise the system uniformly, e.g. with $z_i = 0, 1$ or 2 everywhere. In the animation, these modifications have been effected in modes 'fractalsand' $0, 1$ and 2. For more details on the astonishing features of the BTW model, see [5].

References

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