## *Answers to exercises*

**1.1** *Moments and moment ratio of the cluster number density in $d = 1$.*

(i) In $d = 1$, the cluster number density $n(s,p) = (1-p)^2 p^s$. Thus the $k$th moment $M_k$ of the cluster number density

$$
\begin{aligned}
M_k(p) &= \sum_{s=1}^{\infty} s^k n(s,p) \\
&= \sum_{s=1}^{\infty} s^k (1-p)^2 p^s \\
&= (1-p)^2 \sum_{s=1}^{\infty} s^k p^s \\
&= (1-p)^2 \sum_{s=1}^{\infty} s^k \exp[s \ln(p)] \\
&= (1-p)^2 \sum_{s=1}^{\infty} s^k \exp(-s/s_\xi) && \text{with } s_\xi = -\frac{1}{\ln(p)} \\
&\approx (1-p)^2 \int_1^{\infty} s^k \exp(-s/s_\xi)\, ds, && u = s/s_\xi; du = ds/s_\xi \\
&= (1-p)^2 \int_{1/s_\xi}^{\infty} (u s_\xi)^k \exp(-u)\, s_\xi du \\
&= (1-p)^2 {s_\xi}^{k+1} \int_{1/s_\xi}^{\infty} u^k \exp(-u)\, du \\
&= (1-p)^2 \left(\frac{-1}{\ln(p)}\right)^{k+1} \int_{-\ln(p)}^{\infty} u^k \exp(-u)\, du.
\end{aligned}
\tag{1.1.1}
$$

Letting $p \to p_c^-$, the lower limit of the integral tends to zero (as $p_c = 1$), and the integral becomes the integral representation of the Gamma function. Using the Taylor expansion $\ln(p) = \ln[1-(1-p)] \approx -(1-p)$ for $p \to p_c^-$ we find

$$
\begin{aligned}
M_k &= (1-p)^2 \frac{1}{(1-p)^{k+1}} k! \\
&= k!(p_c - p)^{1-k}
\end{aligned}
\tag{1.1.2}
$$

so we identity $\Gamma_k = k!$ and $\gamma_k = k - 1$.

Alternative derivation with the use of "a trick":

$$\begin{aligned} M_k &= \sum_{s=1}^{\infty} s^k n(s,p) \\ &= (1-p)^2 \sum_{s=1}^{\infty} \left(p\frac{d}{dp}\right)^k p^s \quad \text{the "trick"} \\ &= (1-p)^2 \left(p\frac{d}{dp}\right)^k \sum_{s=1}^{\infty} p^s \\ &= (1-p)^2 \left(p\frac{d}{dp}\right)^k \frac{p}{1-p} \\ &\stackrel{?}{=} k!(1-p)^{1-k} \qquad \text{for } k \geq 2, \end{aligned} \tag{1.1.3}$$

followed by proof by induction.
First the case $k = 2$.

$$\begin{aligned} M_{k=2} &= (1-p)^2 \left(p\frac{d}{dp}\right)^2 \frac{p}{1-p} \\ &= (1-p)^2 \left(p\frac{d}{dp}\right) p\frac{(1-p)\cdot 1 + p}{(1-p)^2} \\ &= (1-p)^2 \left(p\frac{d}{dp}\right) \frac{p}{(1-p)^2} \\ &= (1-p)^2 p\frac{(1-p)^2\cdot 1 + p\cdot 2(1-p)}{(1-p)^4} \\ &= p\frac{(1-p)+2p}{1-p} \\ &= \frac{p+p^2}{1-p} \\ &\to \frac{2}{1-p} \\ &= 2!(p_c-p)^{-1} \qquad \text{for } p \to p_c = 1. \end{aligned} \tag{1.1.4}$$

Now, assume that

$$M_k = (1-p)^2 \left(p\frac{d}{dp}\right)^k \frac{p}{1-p} = k!(1-p)^{1-k} \qquad \text{for } k \geq 2. \tag{1.1.5}$$

Then

$$\begin{aligned} M_{k+1} &= (1-p)^2 \left(p\frac{d}{dp}\right)^{k+1} \frac{p}{1-p} \\ &= (1-p)^2 \left(p\frac{d}{dp}\right)\left(p\frac{d}{dp}\right)^{k} \frac{p}{1-p} \\ &= (1-p)^2 \left(p\frac{d}{dp}\right) k!(1-p)^{-1-k} \quad \text{using Equation (1.1.5)} \\ &= (1-p)^2 pk!(-1-k)(1-p)^{-2-k}\cdot(-1) \\ &= p(k+1)!(1-p)^{-k} \\ &\to (k+1)!(1-p)^{1-(k+1)} \quad \text{for } p \to p_c = 1, \end{aligned} \tag{1.1.6}$$

so the assumption Equation (1.1.5) is true for $k+1$, which completes our proof.

(ii) Note that $M_1 = \sum_{s=1}^{\infty} sn(s,p) = p$ for $p < 1$ so $\Gamma_1 = p$ and $\gamma_1 = 0$. Hence,the moment ratio

$$\begin{aligned} g_k &= \frac{M_k M_1^{k-2}}{M_2^{k-1}} \\ &= \frac{\Gamma_k(1-p)^{1-k}\Gamma_1^{k-2}\left[(1-p)^0\right]^{k-2}}{\Gamma_2^{k-1}\left[(1-p)^{-1}\right]^{k-1}} \\ &= \frac{\Gamma_k\Gamma_1^{k-2}}{\Gamma_2^{k-1}} \end{aligned} \tag{1.1.7}$$

Since $\Gamma_1 \to 1$ for $p \to p_c^-$ we find

$$\begin{aligned} g_k &\to \frac{\Gamma_k}{\Gamma_2^{k-1}} \quad \text{for } p \to p_c^- \\ &= \frac{k!}{2!^{k-1}} \end{aligned} \tag{1.1.8}$$

which is a constant for a given $k$.

**1.2** *Site percolation and site-bond percolation in $d = 1$.*

(i) (a) A percolating (infinite) cluster is present at $p_c$. In one dimension, a percolating cluster can have no empty sites. Therefore $p_c = 1$.

(b) A cluster of size $s$ has $s$ consecutive sites occupied, each with probability $p$, and two empty sites, one at either end, each with probability $(1-p)$, so

$$n(s,p) = p^s(1-p)^2. \qquad (1.2.1)$$

(c) Since $n(s,p)$ is the number of $s$ clusters per lattice site, $sn(s,p)$ is the probability that an arbitrary site belongs to an $s$ cluster. Summing over all possible sizes of clusters, we obtain the probability that an arbitrary site is occupied, that is,

$$\sum_{s=1}^{\infty} sn(s,p) = p \quad \text{for } p < 1. \qquad (1.2.2)$$

This identity is not valid at $p=1$ where the percolating cluster is occupying all the lattice leaving $n(s,p)=0$ for $p=1$.

(d) We find that

$$\begin{aligned}
\sum_{s=1}^{\infty} s^2 p^s &= \sum_{s=1}^{\infty} \left(p\frac{d}{dp}\right)\left(p\frac{d}{dp}\right) p^s \\
&= \left(p\frac{d}{dp}\right)\left(p\frac{d}{dp}\right) \sum_{s=1}^{\infty} p^s \\
&= \left(p\frac{d}{dp}\right)\left(p\frac{d}{dp}\right) \frac{p}{1-p} && \text{for } p<1 \\
&= \left(p\frac{d}{dp}\right) p\frac{(1-p)+p}{(1-p)^2} && \text{for } p<1 \\
&= \left(p\frac{d}{dp}\right) \frac{p}{(1-p)^2} && \text{for } p<1 \\
&= p\frac{(1-p)^2 + p2(1-p)}{(1-p)^4} && \text{for } p<1 \\
&= p\frac{1+p}{(1-p)^3} && \text{for } p<1. \qquad (1.2.3)
\end{aligned}$$

(e) Using the above results we find

$$\chi(p) = \frac{\sum_{s=1}^{\infty} s^2 p^s (1-p)^2}{\sum_{s=1}^{\infty} sn(s,p)} = \frac{(1-p)^2 p \frac{1+p}{(1-p)^3}}{p} = \frac{1+p}{1-p}. \tag{1.2.4}$$

(f) Therefore

$$\chi(p) \to \frac{1+p_c}{1-p} = \frac{2}{p_c - p} = \quad \text{for } p \to p_c^-, \tag{1.2.5}$$

so we identify the amplitude $\Gamma = 2$ and the critical exponent $\gamma = 1$.

(ii) (a) A percolating (infinite) cluster is present at $(p_c, q_c)$. Therefore, no sites nor bonds can be empty, implying $(p_c, q_c) = (1, 1)$.

(b) An $s$ cluster has $s$ consecutive site occupied, each with probability $p$, and $s-1$ consecutive bonds occupied, each with probability $q$. Since $pq$ is the probability to have a site-bond occupied, $(1-pq)^2$ is the probability that a cluster does not continue at either end. Therefore

$$n(s,p,q) = p^s q^{s-1} (1-pq)^2 . \tag{1.2.6}$$

(c) First,

$$\sum_{s=1}^{\infty} sn(s,p,q) = \sum_{s=1}^{\infty} sp^s q^{s-1} (1-pq)^2 = \frac{1}{q} \sum_{s=1}^{\infty} s(pq)^s (1-pq)^2 = \frac{1}{q} pq = p \tag{1.2.7}$$

and similarly

$$\begin{aligned}\sum_{s=1}^{\infty} s^2 n(s,p,q) &= \sum_{s=1}^{\infty} s^2 p^s q^{s-1} (1-pq)^2 \\ &= \frac{1}{q}(1-pq)^2 \sum_{s=1}^{\infty} s^2 (pq)^s \\ &= \frac{1}{q}(1-pq)^2\, pq \frac{1+pq}{(1-pq)^3} \\ &= p\frac{1+pq}{1-pq} \qquad (1.2.8)\end{aligned}$$

so that

$$\chi(p,q) = \frac{1+pq}{1-pq}. \qquad (1.2.9)$$

This result is identical to that of site percolation if we identify the occupation probability with $pq$, that is, a site-bond is the equivalent of a site.

**1.3** *Percolation in $d = 1$ on a lattice with periodic boundary conditions.*

(i) When $s \leq L-2$, an $s$-cluster must be bounded by two empty sites. For $s = L-1$, there is only one empty site in the system while for $s = L$, all sites are occupied. Clearly we cannot have $s > L$. Thus

$$n(s,p) = \begin{cases} p^s(1-p)^2 & \text{for } s \leq L-2 \\ p^{L-1}(1-p) & \text{for } s = L-1 \\ p^L & \text{for } s = L \\ 0 & \text{for } s > L. \end{cases} \tag{1.3.1}$$

(ii) A cluster with $s = L$ is percolating and hence not to be characterized as being finite. Therefore, $\sum_{s=1}^{L-1} sn(s,p)$ represents the probability that a site belongs to a finite cluster.

(iii) In a $d = 1$ system of size $L$, the probability of an arbitrarily selected site to belong to the spanning (infinite) cluster

$$P_\infty(L,p) = p^L. \tag{1.3.2}$$

Alternatively, an occupied site either belongs to the spanning cluster or to a finite cluster ($s < L$), that is,

$$\begin{aligned} P_\infty(L,p) &= p - \sum_{s=1}^{L-1} sn(s,p) \\ &= p - (L-1)p^{L-1}(1-p) - \sum_{s=1}^{L-2} sp^s(1-p)^2 \\ &= p - (L-1)p^{L-1}(1-p) - (1-p)^2 \left(p\frac{d}{dp}\right)\left(\sum_{s=1}^{L-2} p^s\right) \\ &= p - (L-1)p^{L-1}(1-p) - (1-p)^2 \left(p\frac{d}{dp}\right)\left(\frac{p-p^{L-1}}{1-p}\right) \\ &= p - (L-1)p^{L-1}(1-p) - (1-p)^2 p\frac{(1-p)(1-(L-1)p^{L-2}) + (p-p^{L-1})}{(1-p)^2} \\ &= p - (L-1)p^{L-1} + (L-1)p^L - (p-p^2)(1-(L-1)p^{L-2}) - p^2 + p^L \\ &= p^L. \end{aligned} \tag{1.3.3}$$

(iv) (a) In $d = 1$ percolation,

$$\xi = -\frac{1}{\ln p} \Leftrightarrow \ln p = -\frac{1}{\xi} \Leftrightarrow p = \exp\left(-\frac{1}{\xi}\right). \qquad (1.3.4)$$

Thus

$$P_\infty(L, \xi) = p^L = \left[\exp\left(-\frac{1}{\xi}\right)\right]^L = \exp\left(-\frac{L}{\xi}\right). \qquad (1.3.5)$$

(b) Write the order parameter using the scaling form

$$P_\infty(\xi; L) = \exp\left(-\frac{L}{\xi}\right) = \xi^{-\beta/\nu}\mathcal{P}(L/\xi), \qquad (1.3.6)$$

where

$$\beta/\nu = 0 \qquad (1.3.7)$$

and a scaling function

$$\mathcal{P}(x) = \exp(-\frac{L}{\xi}) \propto \begin{cases} \text{constant} & \text{for } L \ll \xi \\ \text{decaying rapidly} & \text{for } L \gg \xi. \end{cases} \qquad (1.3.8)$$

**1.4** *Cluster number density scaling functions in d=1 and the Bethe lattice.*

(i) (a) Rewriting the cluster number density in $d = 1$ we find

$$\begin{aligned} n(s,p) &= (1-p)^2 p^s \\ &= (p_c - p)^2 \exp(-s/s_\xi) \qquad \text{with } s_\xi = -\frac{1}{\ln p} \\ &= s^{-2}[s(p_c - p)]^2 \exp(-s/s_\xi) \\ &\approx s^{-2}\,(s/s_\xi)^2 \exp(-s/s_\xi) \qquad \text{for } p \to p_c^- \\ &= s^{-2}\mathcal{G}_{1d}(s/s_\xi) \end{aligned} \tag{1.4.1}$$

with

$$\mathcal{G}_{1d}(s/s_\xi) = (s/s_\xi)^2 \exp(-s/s_\xi). \tag{1.4.2}$$

and

$$s_\xi \to (p_c - p)^{-1} \quad \text{for } p \to p_c^-. \tag{1.4.3}$$

Thus we identify

$$\tau = 2, \tag{1.4.4a}$$

$$\sigma = 1, \tag{1.4.4b}$$

$$a = 1, \tag{1.4.4c}$$

$$b = 1. \tag{1.4.4d}$$

(b) From the graph of the scaling function $\mathcal{G}_{1d}$, see Figure 1.4.1, we see that for small arguments $s \ll s_\xi$, the function increases quadratically in the argument $s/s_\xi$ while it decays exponentially fast for $s \gg s_\xi$. Indeed, such cluster sizes are exponentially rare as the characteristic cluster size $s_\xi$ is the typical size of the largest cluster.

(c) The scaling function $\mathcal{G}_{1d}(x) = x^2 \exp(-x)$ and

$$\mathcal{G}_{1d}^{(1)}(x) = 2x \exp(-x) - x^2 \exp(-x) = (2x - x^2) \exp(-x)$$
$$\mathcal{G}_{1d}^{(2)}(x) = (2 - 2x - 2x + x^2) \exp(-x) = (2 - 4x + x^2) \exp(-x)$$

Hence $\mathcal{G}_{1d}(0) = \mathcal{G}_{1d}^{(1)}(0) = 0, \mathcal{G}_{1d}^{(2)}(0) = 2$. Thus the Taylor

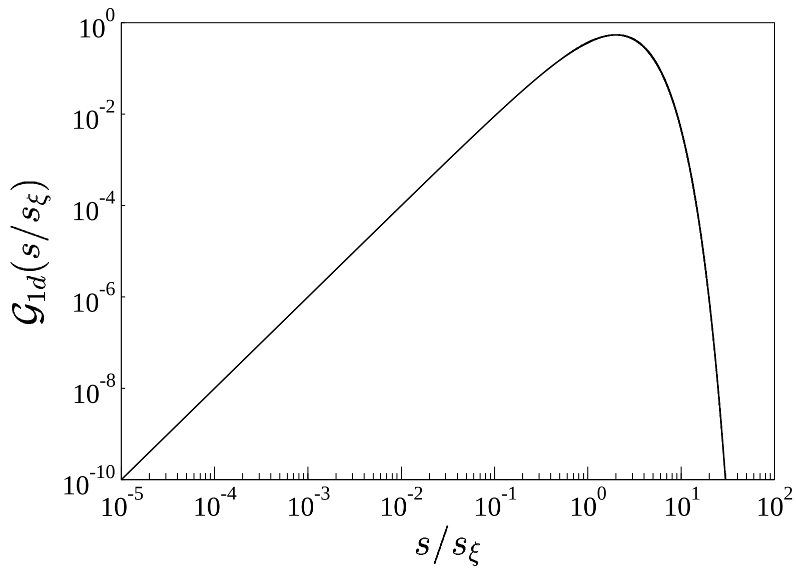



Fig. 1.4.1 The scaling function $\mathcal{G}_{1d}$ in $d = 1$ increases like $(s/s_\xi)^2$ for small arguments and decays (exponentially) fast for large arguments.

expansion of $\mathcal{G}_{1d}$ around zero,

$$\begin{aligned}\mathcal{G}_{1d}(s/s_\xi) &= \mathcal{G}_{1d}(0) + \mathcal{G}_{1d}^{(1)}(0)s/s_\xi + \frac{1}{2}\mathcal{G}_{1d}^{(2)}(0)\,(s/s_\xi)^2 + \cdots \\ &= (s/s_\xi)^2 + \cdots \end{aligned} \qquad (1.4.5)$$

which is consistent with Figure 1.4.1.

(ii) (a) On a Bethe lattice with $z = 3$ where $p_c = 1/2$ we have

$$\begin{aligned} n(s,p) &\propto s^{-5/2}\exp(-s/s_\xi) && s \gg 1 \\ s_\xi &= -\frac{1}{\ln(4p-4p^2)} \to \frac{1}{4}(p-p_c)^{-2} && \text{for } p \to p_c. \end{aligned}$$

Thus we identify the scaling function

$$\mathcal{G}_{\mathrm{Bethe}}(s/s_\xi) = \exp(-s/s_\xi). \qquad (1.4.6)$$

with

$$\tau = 5/2 \qquad (1.4.7a)$$
$$\sigma = 1/2 \qquad (1.4.7b)$$
$$b = 1/4. \qquad (1.4.7c)$$

It would be possible to determine $a$ by applying a normalisation constraint. For example when $p < p_c$ the cluster number density must satisfy

$$\sum_{s=1}^{\infty} sn(s,p) = a \sum_{s=1}^{\infty} s^{1-\tau} \mathcal{G}_{\text{Bethe}}(s/s_\xi) = p. \tag{1.4.8}$$

This constraint will determine $a$.

(b) From the graph of the scaling function $\mathcal{G}_{\text{Bethe}}$, see Figure 1.4.2, we see that for small arguments $s \ll s_\xi$, the function is approximately constant while it decays exponentially fast for $s \gg s_\xi$. Indeed, such cluster sizes are exponentially rare as the characteristic cluster size $s_\xi$ is the typical size of the largest cluster.

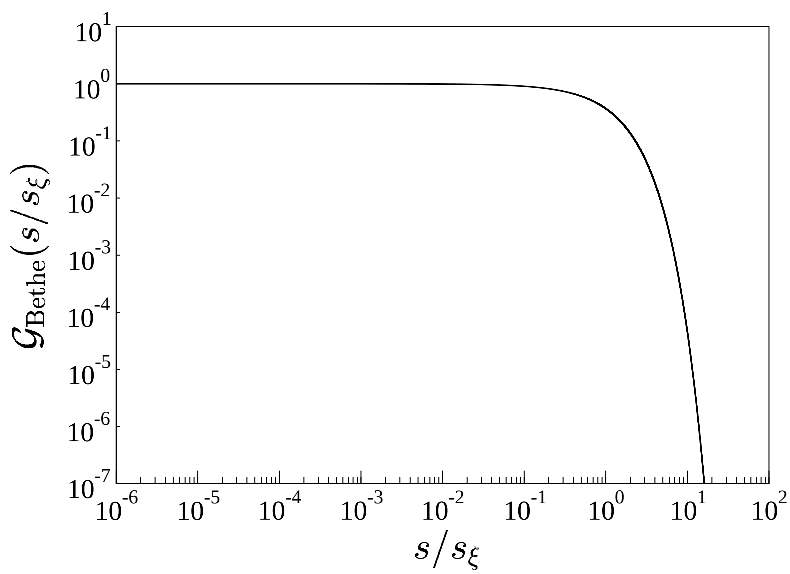



Fig. 1.4.2 The scaling function $\mathcal{G}_{\text{Bethe}}$ for the Bethe lattice is approximately constant for small arguments and decays exponentially fast for large arguments.

(c) Clearly

$$\mathcal{G}_{\text{Bethe}}(x) = 1 - x + \cdots \approx 1, \tag{1.4.9}$$

consistent with Figure 1.4.2.

**1.5** *Moments of the cluster number density.*

(i) We approximate the sum by an integral:

$$\begin{aligned}
M_k(p) &= \sum_{s=1}^{\infty} s^k n(s,p) \\
&= \sum_{s=1}^{\infty} a s^{k-\tau} \mathcal{G}(s/s_\xi) \\
&\approx \int_1^{\infty} a s^{k-\tau} \mathcal{G}(s/s_\xi)\, ds \\
&= \int_{1/s_\xi}^{\infty} a\,(s_\xi u)^{k-\tau}\, \mathcal{G}(u) s_\xi\, du \qquad \text{with } u = s/s_\xi \\
&= s_\xi^{k+1-\tau} a \int_{1/s_\xi}^{\infty} u^{k-\tau} \mathcal{G}(u)\, du \\
&= |p-p_c|^{-(k+1-\tau)/\sigma} a b^{k+1-\tau} \int_0^{\infty} u^{k-\tau} \mathcal{G}(u)\, du \quad \text{for } p \to p_c \\
&= \Gamma_k\, |p-p_c|^{-\gamma_k} \qquad (1.5.1)
\end{aligned}$$

where

$$\gamma_k = \frac{k+1-\tau}{\sigma} \qquad (1.5.2a)$$

$$\Gamma_k = a b^{k+1-\tau} \int_0^{\infty} u^{k-\tau} \mathcal{G}(u)\, du. \qquad (1.5.2b)$$

The critical amplitude $\Gamma_k$ is just a number independent of $p$. Note that we recover the scaling relation

$$\gamma = \frac{3-\tau}{\sigma} \qquad (1.5.3)$$

by letting $k = 2$.

(ii) The moment ratio

$$\begin{aligned}
g_k &= \frac{M_k M_1^{k-2}}{M_2^{k-1}} \\
&= \frac{\Gamma_k \Gamma_1^{k-2}}{\Gamma_2^{k-1}} \qquad (1.5.4) \\
&= \frac{\int_0^{\infty} u^{k-\tau} \mathcal{G}(u)\, du \left[\int_0^{\infty} u^{1-\tau} \mathcal{G}(u)\, du\right]^{k-2}}{\left[\int_0^{\infty} u^{2-\tau} \mathcal{G}(u)\, du\right]^{k-1}}
\end{aligned}$$

(iii) In $d = 1$ percolation, $\tau = 2, \sigma = 1, a = 1, b = 1$ and the scaling function $\mathcal{G}_{1d}(u) = u^2 \exp(-u)$ so

$$\begin{aligned}\Gamma_k &= \int_0^\infty u^k \exp(-u)\, du \\ &= k!\end{aligned}$$

**1.6** *Universality of the ratio of amplitudes for the average cluster size.* By definition

$$\chi(p) = \frac{\sum_{s=1}^\infty s^2 n(s,p)}{\sum_{s=1}^\infty sn(s,p)} \tag{1.6.1}$$

where the denominator $\sum_{s=1}^\infty sn(s,p) = p_c$ at $p = p_c$. Since we are ultimately interested in the limit $p \to p_c$, we simply substitute the denominator with $p_c$.

We thus find

$$\begin{aligned}p_c\chi(p) &= \sum_{s=1}^\infty s^2 n(s,p) \\ &= \sum_{s=1}^\infty a s^{2-\tau} \mathcal{G}_\pm(s/s_\xi) \\ &\approx \int_1^\infty a s^{2-\tau} \mathcal{G}_\pm(s/s_\xi)\, ds\end{aligned} \tag{1.6.2}$$

Substituting $u = s/s_\xi$, that is $s = s_\xi u$ and $ds = s_\xi du$. With the new lower integration limit $1/s_\xi$ we have

$$\begin{aligned}p_c\chi(p) &= \int_{1/s_\xi}^\infty a\,(s_\xi u)^{2-\tau}\, \mathcal{G}_\pm(u) s_\xi\, du \\ &= s_\xi^{3-\tau} a \int_{1/s_\xi}^\infty u^{2-\tau} \mathcal{G}_\pm(u)\, du \\ &= |p-p_c|^{-(3-\tau)/\sigma} a b^{3-\tau} \int_0^\infty u^{2-\tau} \mathcal{G}_\pm(u)\, du \quad \text{for } p \to p_c\end{aligned}$$

where we, in the last step, have substituted $s_\xi = b|p-p_c|^{-1/\sigma}$ for $p \to p_c$ and changed the lower limit to zero as $s_\xi$ diverges at $p = p_c$.

(i) Assume $p < p_c$. Then, in the limit $p \to p_c^-$,

$$\chi(p) = (p_c - p)^{-(3-\tau)/\sigma} \frac{ab^{3-\tau}}{p_c} \int_0^\infty u^{2-\tau} \mathcal{G}_-(u)\, du \tag{1.6.3}$$

with

$$\gamma^- = \frac{3-\tau}{\sigma} \tag{1.6.4a}$$

$$\Gamma^- = \frac{ab^{3-\tau}}{p_c} \int_0^\infty u^{2-\tau} \mathcal{G}_-(u)\, du. \tag{1.6.4b}$$

(ii) Assume $p > p_c$. Then, in the limit $p \to p_c^+$,

$$\chi(p) = (p - p_c)^{-(3-\tau)/\sigma} \frac{ab^{3-\tau}}{p_c} \int_0^\infty u^{2-\tau} \mathcal{G}_+(u)\, du \tag{1.6.5}$$

with

$$\gamma^+ = \frac{3-\tau}{\sigma} \tag{1.6.6a}$$

$$\Gamma^+ = \frac{ab^{3-\tau}}{p_c} \int_0^\infty u^{2-\tau} \mathcal{G}_+(u)\, du. \tag{1.6.6b}$$

(iii) (a) By inspection $\gamma^- = \gamma^+ = (3 - \tau)/\sigma$.

(b) The ratio of critical amplitudes

$$\frac{\Gamma^+}{\Gamma^-} = \frac{\int_0^\infty u^{2-\tau} \mathcal{G}_-(u)\, du}{\int_0^\infty u^{2-\tau} \mathcal{G}_+(u)\, du} \tag{1.6.7}$$

is independent of the proportionality constants $a$ and $b$ and $p_c$ and only depends on the universal critical exponent $\tau$ and the universal scaling functions $\mathcal{G}_\pm$. Thus the ration $\Gamma^+/\Gamma^-$ is itself universal.

(c) The ratio of the critical amplitudes $\Gamma^+/\Gamma^-$ is related to the distance between the numerical results for the average cluster size for $p < p_c$ and $p > p_c$ respectively. Numerical simulations confirm that $\Gamma^+/\Gamma^-$ is universal and one finds $\Gamma^+/\Gamma^- \approx 200$ using the numerical results displayed.

**1.7** *The order parameter on a Bethe lattice with coordination number $z$.*

(i) $P_\infty(p)$ is the probability that an arbitrarily selected site belongs to the percolating infinite cluster. Consider the 'origin' in the Bethe lattice.

$$\begin{aligned} P_\infty(p) &= \text{probability 'origin' is occupied} \cdot \\ &\quad \text{probability } \textit{at least one} \text{ of the } z \text{ branches connects to infinity} \\ &= p\,[1 - Q_\infty^z(p)] \end{aligned} \tag{1.7.1}$$

where $Q_\infty(p)$ denotes the probability that a given branch does *not* connect to infinity. Again, we will rely on the fact that all sites in a Bethe lattice are equivalent, so $Q_\infty(p)$ is also the probability that a subbranch does not connect to infinity. Hence

$$\begin{aligned} Q_\infty(p) &= \text{neighbour to 'origin' is empty } + \text{ neighbour to 'origin' is occupied} \\ &\quad \text{but } \textit{none} \text{ of the } (z-1) \text{ subbranches connect to infinity} \\ &= 1 - p + pQ_\infty^{z-1}(p) \end{aligned} \tag{1.7.2}$$

(ii) For convenience, we drop the $p$-dependence of $Q_\infty(p)$ and simply write $Q_\infty$. Let

$$Q_\infty^{z-1} = (1 - [1 - Q_\infty])^{z-1} = (1-x)^{z-1} \quad \text{with } x = 1 - Q_\infty.$$

We expand to second order in $x$ around $x = 0$.

$$\begin{aligned} f(x) &= (1-x)^{z-1} & &\Rightarrow f(0) = 1 \\ f^{(1)}(x) &= -(z-1)(1-x)^{z-2} & &\Rightarrow f^{(1)}(0) = -(z-1) \\ f^{(2)}(x) &= (z-1)(z-2)(1-x)^{z-3} & &\Rightarrow f^{(2)}(0) = (z-1)(z-2) \end{aligned} \tag{1.7.3}$$

implying

$$\begin{aligned} Q_\infty^{z-1} &\approx f(0) + f^{(1)}(0)x + \frac{1}{2}f^{(0)}(0)x^2 + \ldots \\ &= 1 - (z-1)x + \frac{1}{2}(z-1)(z-2)x^2 + \ldots \\ &= 1 - (z-1)(1-Q_\infty) + \frac{1}{2}(z-1)(z-2)(1-Q_\infty)^2 \end{aligned} \tag{1.7.4}$$

Using the Taylor expansion result in Equation (1.7.2) we find

$$\begin{aligned} Q_\infty &= 1 - p + pQ_\infty^{z-1} \\ &\approx 1 - p + p - p(z-1)(1-Q_\infty) + \overbrace{p\frac{1}{2}(z-1)(z-2)}^{a}(1-Q_\infty)^2 \\ &= 1 - p(z-1) + p(z-1)Q_\infty + a(1 + Q_\infty^2 - 2Q_\infty) \qquad (1.7.5) \end{aligned}$$

and rearranging

$$\begin{aligned} &aQ_\infty^2 + \{\overbrace{p(z-1)-1}^{b} - 2a\}Q_\infty + a + \overbrace{1-p(z-1)}^{-b} = 0 \Leftrightarrow \\ &aQ_\infty^2 + (b-2a)Q_\infty + a - b = 0 \Leftrightarrow \qquad (1.7.6) \\ &Q_\infty = \frac{2a - b \pm \sqrt{(b-2a)^2 - 4a(a-b)}}{2a} = \frac{2a - b \pm \sqrt{b^2}}{2a}. \end{aligned}$$

As $b > 0$ since $p$ eventually is larger that $\frac{1}{z-1}$ we find

$$Q_\infty = \begin{cases} 1 & \text{for } p \le p_c \\ \frac{a-b}{a} & \text{for } p > p_c. \end{cases}$$

The solution $Q_\infty = 1 \Rightarrow P_\infty(p) = 0$ belongs to the regime $p \le p_c$. The other solution is nontrivial and belongs to the regime $p > p_c$, and hence

$$Q_\infty(p) = \begin{cases} 1 & \text{for } p \le p_c \\ 1 - \frac{2p(z-1)-2}{p(z-1)(z-2)} & \text{for } p > p_c. \end{cases} \qquad (1.7.7)$$

(iii) The relevant solution has $Q_\infty < 1$. Substituting into the Equation (1.7.1)

$$\begin{aligned} P_\infty(p) &= p(1 - Q_\infty^z) \\ &= p\left[1 - \left(1 - \frac{2p(z-1)-2}{p(z-1)(z-2)}\right)^z\right] \\ &= p\left[1 - \left(1 - \frac{b}{a}\right)^z\right] \\ &= p - p(1 - \frac{b}{a})^z. \qquad (1.7.8) \end{aligned}$$

Note that the ratio

$$\frac{b}{a} = \frac{p(z-1)-1}{p\frac{1}{2}(z-1)(z-2)} \to 0 \qquad \text{for } p \to \tfrac{1}{z-1} = p_c,$$

so $b/a$ is a small quantity for $p \to p_c$. Let $g(x) = (1-x)^z$. Taylor expanding to first order we find $g(0) = 1, g^{(1)}(x) = -z(1-x)^{z-1}, g^{(1)}(0) = -z$ so $(1-x)^z \approx 1 - zx$ for $x \to 0$. Thus

$$\begin{aligned} P_\infty(p) &= p - p\left(1 - \frac{b}{a}\right)^z \\ &\approx p - p\left(1 - z\frac{b}{a}\right) \quad \text{for } p \to p_c \\ &= pz\frac{b}{a} \\ &= pz\frac{2p(z-1)-2}{p(z-1)(z-2)} \\ &= \frac{2z}{z-2}\left(p - \frac{1}{z-1}\right) \\ &= \frac{2z}{z-2}(p - p_c) \end{aligned} \tag{1.7.9}$$

with

$$p_c = \frac{1}{z-1}$$

and

$$A = \frac{2z}{z-2}.$$

Therefore, when $z = 3$ we have $p_c = 1/2$ and $A = 6$.

**1.8** *Finite-size scaling and scaling function for the average cluster size.*

(i) The average cluster size is by definition

$$\chi(p; L = \infty) = \frac{\sum_{s=1}^{\infty} s^2 n(s,p)}{\sum_{s=1}^{\infty} s n(s,p)}. \tag{1.8.1}$$

For $p \to p_c$, the denominator approaches the constant $p_c$. Substituting the sum with an integral we find

$$\begin{aligned}
\chi(p; L = \infty) &\propto \int_1^{\infty} s^2 n(s,p)\, ds \\
&= \int_1^{\infty} s^{2-\tau} \mathcal{G}\left(s/s_\xi\right)\, ds \\
&= \int_{1/s_\xi}^{\infty} (u s_\xi)^{2-\tau} \mathcal{G}(u) s_\xi\, du \\
&= s_\xi^{3-\tau} \int_{1/s_\xi}^{\infty} u^{2-\tau} \mathcal{G}(u)\, du \\
&\propto |p - p_c|^{-(3-\tau)/\sigma}
\end{aligned} \tag{1.8.2}$$

as for $p \to p_c$, $s_\xi \to \infty$ and the integral approaches a constant number. Thus

$$\gamma = \frac{3-\tau}{\sigma}. \tag{1.8.3}$$

(ii) For $p \to p_c$, the correlation length

$$\xi(p) \propto |p - p_c|^{-\nu} \Rightarrow |p - p_c| \propto \xi^{-\frac{1}{\nu}}, \tag{1.8.4}$$

that is,

$$\chi(\xi; L = \infty) \propto |p - p_c|^{-\gamma} \propto \xi^{\gamma/\nu} \quad \text{for } p \to p_c. \tag{1.8.5}$$

(iii) (a) There are only two relevant length scales in the problem, the correlation length $\xi$ and the lattice size $L$. When $L \ll \xi$, $L$ will be the limiting length scale taking the role of $\xi$ and thus

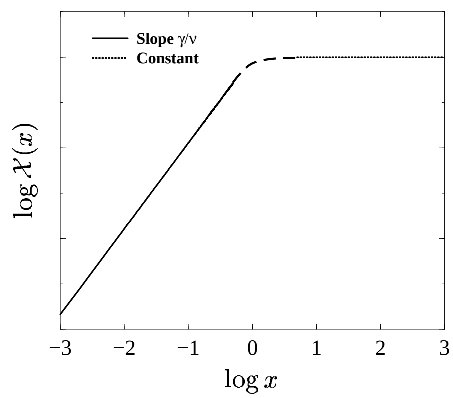
$$\chi(\xi; L) \propto L^{\gamma/\nu} \quad \text{for } p \to p_c, 1 \ll L \ll \xi. \tag{1.8.6}$$

(b) When $L \gg \xi$, we 'do not know' that the lattice is finite and $\chi(\xi; L)$ will be independent of the lattice size $L$. Thus we have

$$\chi(\xi; L) = \begin{cases} \xi^{\gamma/\nu} & \text{for } L \gg \xi \\ L^{\gamma/\nu} & \text{for } L \ll \xi \end{cases}$$
$$= \xi^{\gamma/\nu}\mathcal{X}(L/\xi) \tag{1.8.7}$$

where the scaling function

$$\mathcal{X}(x) = \begin{cases} constant & \text{for } x \gg 1 \\ x^{\gamma/\nu} & \text{for } x \ll 1. \end{cases} \tag{1.8.8}$$




(c) If $p = p_c$, the correlation length $\xi = \infty$ and we are always in the case $L \ll \xi$ $(x \ll 1)$. Thus by plotting $\log \chi(L, \infty)$ versus $\log L$ we get a straight line with slope $\gamma/\nu$.

(iv) At $p = p_c, \xi = \infty$ so according to Equation (1.8.6) the average cluster size $\chi(\xi = \infty; L) \propto L^{\gamma/\nu}$. Hence we find

$$\begin{aligned} \chi(\xi = \infty; L) &\propto \int_1^\infty s^2 n(s, p_c, L)\, ds \\ &= \int_1^\infty s^{2-\tau}\tilde{\mathcal{G}}\left(s/L^D\right)\, ds \\ &= \int_{1/L^D}^\infty (uL^D)^{2-\tau}\tilde{\mathcal{G}}(u)L^D\, du \\ &= L^{D(3-\tau)}\int_{1/L^D}^\infty u^{2-\tau}\tilde{\mathcal{G}}(u)\, du \\ &\propto L^{D(3-\tau)} \end{aligned} \tag{1.8.9}$$

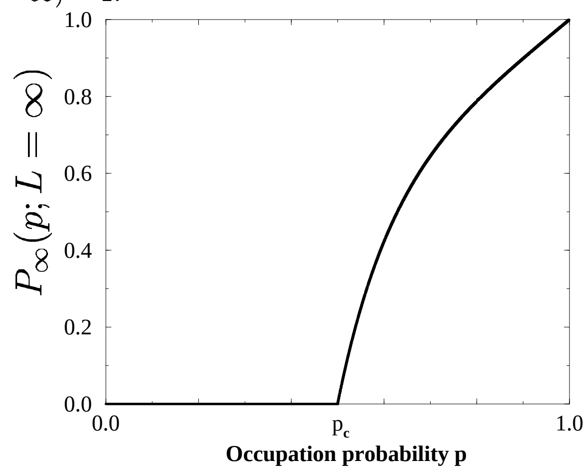
since the integral approaches a number when $L \gg 1$. Thus

$$\frac{\gamma}{\nu} = D(3 - \tau). \tag{1.8.10}$$

**1.9** *Finite-size scaling and scaling function for the order parameter.*

(i) (a) The order parameter $P_\infty(p; L = \infty)$ is the probability that (at occupation probability $p$) an arbitrary site belongs to the percolating infinite cluster.

(b) For $p \leq p_c$ there are no percolating infinite clusters, so $P_\infty(p; L = \infty) = 0$. The critical occupation probability is the concentration $p_c$ above which a percolating infinite cluster appears for the first time and the order parameter becomes nonzero for $p > p_c$. Clearly $P_\infty(p = 1; L = \infty) = 1$.




(c) The probability that an arbitrary site belongs to an $s$-cluster is $sn(s, p; L = \infty)$. The probability that an arbitrary site belongs to any finite cluster is $\sum_{s=1}^{\infty} sn(s, p; L = \infty)$. The relation thus states that for a given site

$$P(\text{in infinite cluster}) = P(\text{occupied}) - P(\text{in finite cluster})$$

$$P_\infty(p; L = \infty) = p - \sum_{s=1}^{\infty} sn(s, p; L = \infty).$$

(ii) (a) For $p \to p_c^+$, the correlation length

$$\xi(p) \propto (p - p_c)^{-\nu} \Rightarrow (p - p_c) \propto \xi^{-\frac{1}{\nu}}, \quad (1.9.1)$$

that is,

$$P_\infty(\xi; L = \infty) \propto (p - p_c)^\beta \propto \xi^{-\beta/\nu}. \quad (1.9.2)$$

(b) There are only two relevant length scales in the problem, the correlation length $\xi$ and the lattice size $L$. When $1 \ll L \ll \xi$, $L$ will be the limiting length scale taking the role of $\xi$ and thus

$$P_\infty(\xi; L) \propto L^{-\beta/\nu} \quad \text{for } 1 \ll L \ll \xi, p \to p_c. \qquad (1.9.3)$$

(c) If $p = p_c$, the correlation length $\xi = \infty$ and we are always in the case $L \ll \xi$ ($x \ll 1$). Thus by plotting $\log P_\infty(\infty; L)$ versus $\log L$ we get a straight line with slope $-\beta/\nu$.

(iii) (a) The order parameter at $p = p_c$ in an infinite lattice is zero. In the limit $L \to \infty$, the cluster number density $n(s, p_c; L)$ tends to $s^{-\tau} g(0)$, so also the right hand side equals zero.

(b) At $p = p_c, \xi = \infty$ so according to Equation (1.9.3) the order parameter $P_\infty(\xi = \infty, L) \propto L^{-\beta/\nu}$. First, we use Equation (1.95) in the question and substitute the scaling law for the cluster number density. Since the main contribution to the sum are from $s \gg 1$, we can replace the sum by an integral and use the substitution $u = s/L^D$ to find

$$\begin{aligned} P_\infty(\xi = \infty; L) &= \sum_{s=1}^{\infty} s^{1-\tau}[g(0) - g(s/L^D)] \\ &\propto \int_1^\infty s^{1-\tau}[g(0) - g(s/L^D)]\, ds \\ &= \int_{1/L^D}^\infty (uL^D)^{1-\tau}[g(0) - g(u)] L^D\, du \\ &= L^{D(2-\tau)} \int_{1/L^D}^\infty u^{1-\tau}[g(0) - g(u)]\, du \\ &\propto L^{D(2-\tau)} \qquad (1.9.4) \end{aligned}$$

as for $L \gg 1$ the integral approaches a number (lower limit approaches zero). Thus

$$P_\infty(\xi = \infty; L) \propto L^{-\beta/\nu} \propto L^{D(2-\tau)} \quad \text{for } L \gg 1, \quad (1.9.5)$$

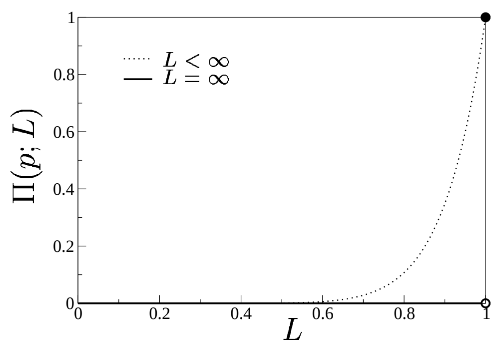
implying the scaling relation

$$-\frac{\beta}{\nu} = D(2-\tau). \tag{1.9.6}$$

**1.10** *Probability of having a percolating cluster on a lattice of size L.*

(i) Since for $p < p_c$ there is no percolating cluster, we have

$$\Pi_\infty(p, L=\infty) = \begin{cases} 0 & \text{for } p < p_c \\ 1 & \text{for } p = p_c. \end{cases}$$




(ii) (a) There is a percolating cluster only if all $L$ sites are occupied. Therefore,

$$\Pi_\infty(p; L) = p^L,$$

see Figure above.

(b) We have

$$\Pi_\infty(\xi; L) = p^L = \exp\left(\ln p^L\right) = \exp\left(L \ln p\right) = \exp\left(-L/\xi\right)$$

using $\xi = -1/\ln p$.

(c) When $p \to p_c^-$, the correlation length $\xi \to (p_c - p)^{-1}$ so $1/\xi = (p_c - p)$. Therefore,

$$\Pi_\infty(\xi; L) \to \exp\left(-(p_c - p)L\right) = \mathcal{F}_{1\mathrm{d}}\left[(p_c - p)L\right] \quad \text{for } p \to p_c^-,$$

where we identify the scaling function $\mathcal{F}_{1\mathrm{d}}(x) = \exp(-x)$. Hence

$$\mathcal{F}_{1\mathrm{d}}(x) = \begin{cases} \text{constant} & \text{for } x \ll 1 \\ \text{decay rapidly} & \text{for } x \gg 1. \end{cases}$$

(iii) We assume that

$$\Pi_\infty(p;L) = \mathcal{G}\left(L/\xi\right) \quad \text{for } p \to p_c,$$

(a) Since $\xi \propto |p-p_c|^{-\nu}$ we have

$$\Pi_\infty(p;L) = \mathcal{G}\left(L/\xi\right) = \mathcal{G}\left(L|p-p_c|^\nu\right) = \mathcal{F}\left(L^{1/\nu}|p-p_c|\right) \text{ for } p \to p_c,$$

so that $\mathcal{F}(x) = \mathcal{G}(x^{1/\nu})$.

(b) In higher dimension, $\Pi_\infty(p;L)$ will approach a step function at $p = p_c$ when $L \to \infty$. Hence, the limiting function $d\Pi_\infty/dp = \delta(p_c - p)$ is a delta-function at $p = p_c$ when $L \to \infty$.

**1.11** *Real-space RG transformation on a square lattice.*

(i) The real space renormalisation technique is based on a so-called *block site technique* and has three basic steps:

**1.** *Divide the lattice into blocks of linear size b.*

**2.** *Next, the coarse graining procedure takes place. The sites in the blocks are* averaged in some way *and the* **entire** *block is replaced by a single (super) site which is occupied with a probability according to the renormalisation group transformation* $R_b$.

Important to keep the symmetry of the original lattice such that the coarse graining procedure can be repeated. The two operations create a new lattice with lattice constant $b$ times as large as in the original lattice.

**3.** *Restore original lattice constant by rescaling length scales by the factor b.*

The coarse graining procedure in step 2 eliminates fluctuations on scales smaller than the block size $b$, therby exploring the large scale behaviour of the system upon iteration.

(ii) It is always a good idea to draw the situation!

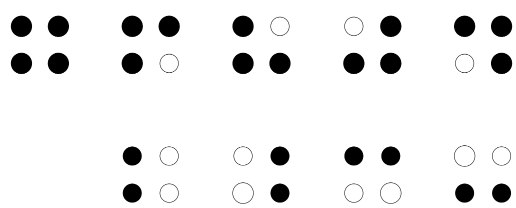


Fig. 1.11.1 The block of size 2×2 contains a spanning cluster if all four sites are occupied – probability $p^4$ – or three sites are occupied, one empty – probability $4p^3(1-p)$ – as there are four different ways of placing the empty site. Also, four different configucations contain a spanning cluster if two sites are occupied and two sites empty – probability $4p^2(1-p)^2$.

Thus

$$\begin{aligned} R_b(p) &= p^4 + 4p^3(1-p) + 4p^2(1-p)^2 \\ &= p^4 - 4p^3 + 4p^2. \end{aligned} \tag{1.11.1}$$

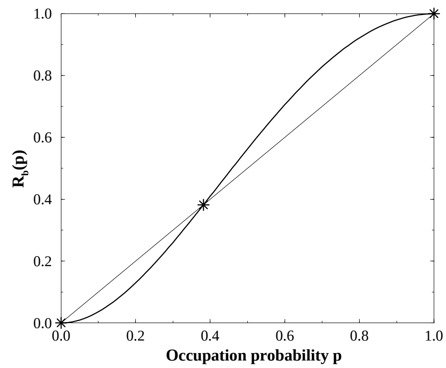



Fig. 1.11.2 The intersection between the graph of the renormalisation group transformation $R_b(p)$ and the identity transformation $R_b(p) = p$ are solutions to the fixed point equation.

(iii) Solving the equation graphically yields

$$p^\star = \begin{cases} 0 & \text{trivial fixed point} \\ 1 & \text{trivial fixed point} \\ 0.38 & \text{non-trivial fixed point.} \end{cases}$$

The correlation lengths $\xi = 0$ for the trivial fixed points $p^\star = 0$ and $p^\star = 1$ correspond to the empty and fully occupied lattice, respectively.

The non-trivial fixed point $p^\star = 0.38$ is accociated with the critical occupation probability $p_c$ where the correlation length is infinite.

When performing the real space renormalisation procedure, length scales are rescaled by the factor $b$. Thus, the correlation length $\xi \to \xi/b$ only remains invariant if $\xi = 0$, associated with the trivial fixed points or $\xi = \infty$, associated with the non-trivial fixed point. Since $\xi \propto |p - p_c|^{-\nu}$, the non-trivial fixed point is identified as $p_c$.

(iv) We identify $p_c = p^\star = 0.38$ (see (iii)). Let A be a constant. Then

$$\xi = A\,|p - p_c|^{-\nu} \tag{1.11.2a}$$

$$\xi' = A\,|R_b(p) - p_c|^{-\nu}. \tag{1.11.2b}$$

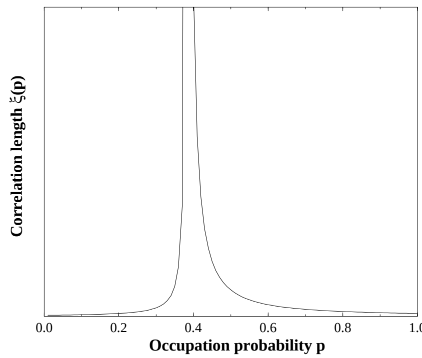



Fig. 1.11.3 If we start out with a finite correlation length, the rescaled correlation length $\xi' = \xi/b$ will decrease ($b > 1$) with an associated flow in $p$-space. Starting out with $0 < p < p_c$, the flow will be toward the trivial fixed point $p = 0$ where $\xi = 0$. If we start out with $1 > p > p_c$, the flow will be toward the trivial fixed point $p = 1$ where $\xi = 0$. If we start at the nontrivial fixed point $p^\star$, there is no flow since $\xi = \infty$, an $p$ remains at $p^\star$ under iterations.

As $\xi' = \xi/b$ we find

$$\begin{aligned} |p - p_c|^{-\nu} &= b|R_b(p) - p_c|^{-\nu} \\ &= b|R_b(p) - R(p_c)|^{-\nu}, \end{aligned} \tag{1.11.3}$$

from which we find for $p \to p_c$

$$\nu = \frac{\log b}{\log\left(\frac{dR_b(p_c)}{dp}\right)}. \tag{1.11.4}$$

Now

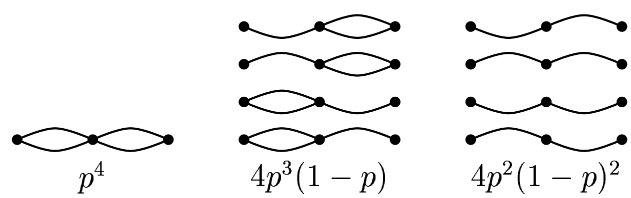
$$\begin{aligned} \frac{dR_b(p_c)}{dp} &= (4p^3 - 12p^2 + 8p)|_{p^\star=0.38} = 1.53 \Rightarrow \\ \nu &= \frac{\log 2}{\log 1.53} = 1.63. \end{aligned} \tag{1.11.5}$$

The exact values in $d = 2$ are $p_c = 0.592746\ldots$ and $\nu = 4/3$, respectively. The discrepancy is due to the approximate nature of the real space renormalisation procedure considered above. For example, the procedure may split a cluster into two or more clusters which must affect the final results we obtain. In general, one would have to introduce a hierarchy of probabilities for each renormalisation step in order to retain the exact properties of the original system. However, since we are essentially only interested in the large scale features of the system, we truncate the hierarchy of probabilities and consider only a single parameter, namely the occupation probability $p$ of a single site.

(v) Quantities which are universal are independent of the microscopic details such as the underlying lattice structure and depend only one the dimensionality of the problem at hand. Examples are the critical exponents, such as $\tau, \nu$, and $\sigma$, describing the behaviour of quantities close to the phase transition and scaling functions. The critical exponents and scaling functions are determined by the large scale properties of the system. Unversality encapsulates the idea that various systems (e.g., site or bond percolation on different underlying lattices) share the same large scale properties.
Non-universal quantities will depend on the lattice structure as e.g. the critical occupation probability $p_c$.

**1.12** *Real-space renormalisation group transformation on a square lattice.*

(i) There are nine configurations that have a connected path from **A** to **B**:




Adding the probabilities for these configurations, we find

$$\begin{aligned} R_b(p) &= p^4 + 4p^3(1-p) + 4p^2(1-p)^2 \\ &= p^4 - 4p^3 + 4p^2. \end{aligned} \tag{1.12.1}$$

(ii) (a) The fixed point Equation $R_b(p) = p$ is solved graphically by plotting the graph of $R_b(p)$ versus $p$ and locating the intersections with the line $R_b(p) = p$.
By inspection, we find the three fixed points

$$p^\star = \begin{cases} 0 & \text{trivial fixed point - empty lattice} \\ 0.38 & \text{non-trivial fixed point} \\ 1 & \text{trivial fixed point - fully occupied lattice.} \end{cases}$$

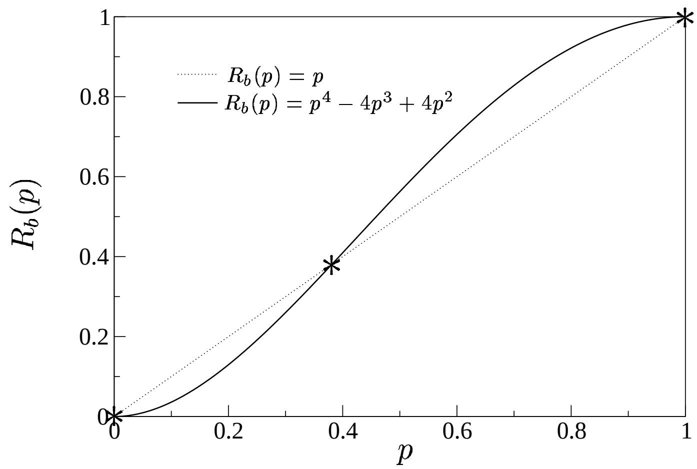



Fig. 1.12.1 The fixed point Equation $R_b(p^\star) = p^\star$ are $p^\star = 0, 0.38, 1$.

(b) When performing the real-space renormalisation procedure, length scales are rescaled by the factor $b$.
If we start out with a finite correlation length, the rescaled correlation length $\xi' = \xi/b$ will decrease ($b > 1$) with an associated flow in $p$-space as indicated below. Starting out with $p < p^\star$, the flow will be toward $p^\star = 0$. If we started out with $p > p^\star$, the flow will be toward $p^\star = 1$.

(c) The correlation length $\xi \to \xi/b$ only remains invariant if $\xi = 0$, associated with the trivial fixed points $p^\star = 0$ (empty lattice) or $p^\star = 1$ (fully occupied lattice) or $\xi = \infty$, associated with the non-trivial fixed point $p^\star \approx 0.38$. Since the correlation length is $\xi = 0$ or $\xi = \infty$ at the fixed point, there is no characteristic scale and scale invariance prevails.

(iii) (a) Let A denote a constant. Then

$$\xi = A\,|p - p_c|^{-\nu} \tag{1.12.2a}$$
$$\xi' = A\,|R_b(p) - p_c|^{-\nu}. \tag{1.12.2b}$$

As $\xi' = \xi/b$ we find

$$|p - p_c|^{-\nu} = b|R_b(p) - p_c|^{-\nu} = b|R_b(p) - R(p_c)|^{-\nu},$$

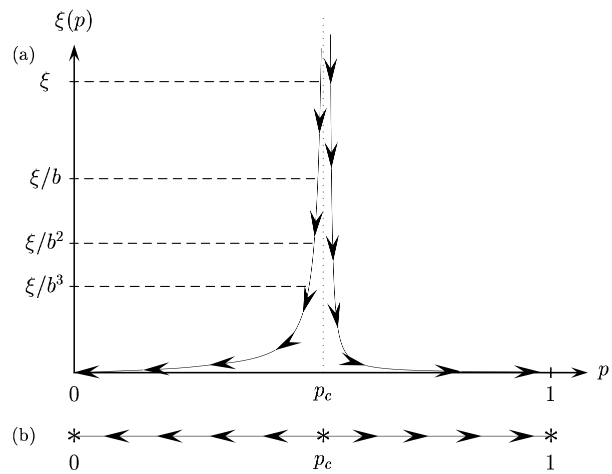



Fig. 1.12.2 (a) A sketch of the correlation length as a function of occupation probability. The dotted line shows the position of $p_c$. (b) The corresponding flow in parameter space.

from which we find for $p \to p_c$

$$\nu = \frac{\log b}{\log\left(\frac{dR_b(p_c)}{dp}\right)}.$$

(b) Now

$$\frac{dR_b}{dp}|_{p^\star} = (4p^3 - 12p^2 + 8p)|_{p^\star=0.38}$$
$$\approx 1.53 \qquad (1.12.3)$$

and hence

$$\nu = \frac{\log 2}{\log 1.53} \approx 1.63. \qquad (1.12.4)$$

The exact values in $d = 2$ are $p_c = 0.5$ and $\nu = 4/3$,

**1.13** *Renormalisation and finite-size scaling of the cluster no. density.*

(i) The square of the radius of gyration $R^2(s)$ for a given $s$-cluster is defined as the average square distance to the centre of mass,

$$R^2(s) = \frac{1}{s}\sum_{i=1}^{s} |r_i - r_{cm}|^2,$$

where $r_i$ denotes the position of the $i$th-particle and $r_{cm}$ the centre of mass. The radius of gyration $R_s^2$ is the average of $R^2(s)$ over all $s$-clusters, that is,

$$R_s = \sqrt{\langle R^2(s) \rangle}. \tag{1.13.1}$$

The radius of gyration $R_s$ measures the linear extent of an $s$-cluster. Thus if $\ell \gg R_s$, the finite cluster is contained within the box of size $\ell$ implying $M(\ell, R_s) = s$. If $\ell \ll R_s$, it appears as if the cluster in infinite (we don't know it is finite). At $p = p_c$, the cluster is fractal with $M(\ell, R_s) \propto \ell^D$, $D$ being the fractal dimension of the infinite percolating cluster. Thus

$$M(\ell, R_s) \propto \begin{cases} \ell^D & \text{for } \ell \ll R_s, \\ s \propto R_s^D & \text{for } \ell \gg R_s \end{cases} \tag{1.13.2}$$

since the mass of the infinite cluster at $p = p_c$ is proportional to $\ell^D$, it is natural to assume that also $s \propto R_s^D$. Thus

$$M(\ell, R_s) = \ell^D m(\ell/R_s), \tag{1.13.3}$$

whit a crossover function

$$m(x) \propto \begin{cases} \text{constant} & \text{for } x \ll 1 \\ x^{-D} & \text{for } x \gg 1 \end{cases} \tag{1.13.4}$$

that is, $D_1 = D$ and $D_2 = 1$.

(ii) From above, we have $M(\ell, R_s) = R_s^D$ for $\ell \gg R_s$. The real space renormalisation transformation renormalises all length scales by a factor $b$, e.g., $R_s \to R_s/b$. Thus

$$s' = M(\ell/b, R_s/b) = (R_s/b)^D = R_s^D/b^D = s/b^D \tag{1.13.5}$$

where we have used $\ell \gg R_s \Rightarrow \ell/b \gg R_s/b$.

(iii) $sn(s, p_c; L)$ is the probability that a site belongs to a cluster of size $s$ in a lattice of linear size $L$ *per lattice site* while $s'n(s', p_c; L/b)$ is the probability that a site belongs to a cluster of size $s'$ in a lattice of linear size $L/b$ *per lattice site*.
As the number of clusters in the original and renormalised lattice is the same we have

$$\begin{aligned} L^d sn(s, p_c; L) &= (L/b)^d s'n(s', p_c; L/b) \Rightarrow \\ sn(s, p_c; L) &= b^{-d} s'n(s', p_c; L/b), \end{aligned} \tag{1.13.6}$$

with $s' = s/b^D$, see question (ii).

(iv) Given the scaling form of the cluster number density

$$n(s, p) = s^{-\tau} \mathcal{G}(s/s_\xi) \quad \text{for } p \to p_c, s \gg 1. \tag{1.13.7}$$

As the characteristic cluster size $s_\xi \propto \xi^D$ where the correlation length $\xi \propto |p - p_c|^{-\nu}$ we find

$$n(s, p) = s^{-\tau} \mathcal{G}(s/\xi^D). \tag{1.13.8}$$

In a finite system at $p = p_c$ where $L \ll \xi = \infty$, one would thus, using a finite-size scaling argument, expect

$$n(s, p) = s^{-\tau} \tilde{\mathcal{G}}(s/L^D). \tag{1.13.9}$$

For $s/L^D \ll 1 \Leftrightarrow s \ll L^D$ (i.e. $L \to \infty$), the cluster number density must be independent of system size, leaving

$$n(s, p_c) \propto s^{-\tau} \Rightarrow \tilde{\mathcal{G}}(x) = \text{constant} \quad x \ll 1. \tag{1.13.10}$$

Clearly $s/L^D \gg 1 \Leftrightarrow s \gg L^D$ is very unlikely, so $\tilde{\mathcal{G}}(x)$ decays rapidly for $x \gg 1$.

(v) Combining Equations (1.13.8) (1.13.6) and (1.13.5) we find

$$\begin{aligned} s^{1-\tau} \mathcal{G}(s/L^D) &= b^{-d} (s/b^D)(s/b^D)^{-\tau} \mathcal{G}\left(\frac{s/b^D}{(L/b)^D}\right) \\ &= b^{-d-D+D\tau} s^{1-\tau} \mathcal{G}(s/L^D), \end{aligned} \tag{1.13.11}$$

from which we conclude

$$-d - D + D\tau = 0, \tag{1.13.12}$$

implying the scaling relation

$$\tau = \frac{d + D}{D}. \tag{1.13.13}$$

***Exercises***

**2.1** *The entropy and the free energy.*

(i) According to the Boltzmann's distribution, the probability $p_r$ to find an equilibrium system in a microstate $r$ with energy $E_r$ at temperature $T$ is given by

$$p_r = \frac{\exp(-\beta E_r)}{\sum_r \exp(-\beta E_r)} = \frac{1}{Z}\exp(-\beta E_r) \tag{2.1.1}$$

where $\beta = 1/(k_B T)$ and $Z$ denotes the partition function. Therefore, the entropy

$$\begin{aligned} S &= -k_B \sum_r p_r \ln p_r \\ &= -k_B \sum_r \frac{1}{Z}\exp(-\beta E_r)\left[\ln\left(\exp(-\beta E_r)\right) - \ln Z\right] \\ &= k_B \ln Z - k_B \sum_r \frac{(-\beta E_r)\exp(-\beta E_r)}{Z} \\ &= k_B \ln Z + \frac{1}{T}\sum_r \frac{E_r \exp(-\beta E_r)}{Z} \\ &= k_B \ln Z + \frac{\langle E\rangle}{T}. \end{aligned}$$

(ii) From part (i) we find

$$\ln Z = \frac{1}{k_B}\left(S - \frac{\langle E\rangle}{T}\right), \tag{2.1.2}$$

so the free energy

$$F = -k_B T \ln Z = -T\left(S - \frac{\langle E\rangle}{T}\right) = \langle E\rangle - TS. \tag{2.1.3}$$

**2.2** *Fluctuation-dissipation theorem.*
First we note that the average total energy

$$\langle E\rangle = -\left(\frac{\partial \ln Z}{\partial \beta}\right)_H, \tag{2.2.1}$$

since

$$-\left(\frac{\partial \ln Z}{\partial \beta}\right)_H = -\frac{1}{Z}\frac{\partial Z}{\partial \beta}$$
$$= -\frac{1}{Z}\frac{\partial}{\partial \beta}\left(\sum_{\{s_i\}} \exp(-\beta E_{\{s_i\}})\right)$$
$$= \frac{1}{Z}\sum_{\{s_i\}} \exp(-\beta E_{\{s_i\}}) E_{\{s_i\}} \tag{2.2.2}$$

However, the *instantaneous* total energy will, of course, fluctuate around the average total energy. The magnitude of the fluctuations is determined by the standard deviation $\Delta E$ where

$$(\Delta E)^2 = \langle (E - \langle E \rangle)^2 \rangle = \langle E^2 + \langle E \rangle^2 - 2E\langle E \rangle \rangle = \langle E^2 \rangle - \langle E \rangle^2.$$

Differentiating twice $\ln Z$ with respect to $\beta$ we find

$$\left(\frac{\partial^2 \ln Z}{\partial \beta^2}\right)_H = -\frac{\partial}{\partial \beta}\left(-\frac{\partial \ln Z}{\partial \beta}\right)_H$$
$$= -\frac{\partial}{\partial \beta}\left(\frac{1}{Z}\sum_{\{s_i\}} \exp(-\beta E_{\{s_i\}}) E_{\{s_i\}}\right)_H$$
$$= \frac{1}{Z}\sum_{\{s_i\}} \exp(-\beta E_{\{s_i\}}) E^2_{\{s_i\}} + \frac{1}{Z^2}\left(\frac{\partial Z}{\partial \beta}\right)_H \sum_{\{s_i\}} \exp(-\beta E_{\{s_i\}}) E_{\{s_i\}}$$
$$= \frac{1}{Z}\sum_{\{s_i\}} \exp(-\beta E_{\{s_i\}}) E^2_{\{s_i\}} + \left(\frac{\partial \ln Z}{\partial \beta}\right)_H \frac{1}{Z}\sum_{\{s_i\}} \exp(-\beta E_{\{s_i\}}) E_{\{s_i\}}$$
$$= \langle E^2 \rangle - \langle E \rangle^2. \tag{2.2.3}$$

However,

$$\left(\frac{\partial^2 \ln Z}{\partial \beta^2}\right)_H = -\left(\frac{\partial \langle E \rangle}{\partial \beta}\right)_H = -\left(\frac{\partial \langle E \rangle}{\partial T}\right)_H \frac{\partial T}{\partial \beta} = -C\frac{\partial (k_B \beta)^{-1}}{\partial \beta} = k_B T^2 C,$$

where $C$ denotes the heat capacity at constant external parameter, such that

$$k_B T^2 C = \langle E^2 \rangle - \langle E \rangle^2. \tag{2.2.4}$$

**2.3** *Eigenvalues, eigenvectors and diagonalisation.*

(i) Assume $\mathbf{x} \neq \mathbf{0}$ is an eigenvector for $f$ with eigenvalue $\lambda$, that is

$$f(\mathbf{x}) = \lambda \mathbf{x}. \tag{2.3.1}$$

Since $f$ is linear,

$$f(\alpha \mathbf{x}) = \alpha f(\mathbf{x}) = \alpha \lambda \mathbf{x} = \lambda(\alpha \mathbf{x}) \tag{2.3.2}$$

so $\alpha \mathbf{x}$ is also an eigenvector with the same eigenvalue $\lambda$.

(ii) Assume $\mathbf{x} \neq \mathbf{0}$ is an eigenvector for $f$ with eigenvalue $\lambda$. If $\mathbf{A}$ is the associated matrix for the linear function $f$ then

$$\mathbf{A}\mathbf{x} - \lambda \mathbf{x} = (\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}, \tag{2.3.3}$$

where $\mathbf{I}$ is the identity matrix. If $\det(\mathbf{A} - \lambda \mathbf{I}) \neq 0$ the matrix $\mathbf{A} - \lambda \mathbf{I}$ would be invertible and the only solution to the Equation (2.3.3) would be the trivial solution $\mathbf{x} = \mathbf{0}$. Equation (2.3.3) can only have non-trivial solutions $\mathbf{x} \neq \mathbf{0}$ if the matrix $\mathbf{A} - \lambda \mathbf{I}$ is not invertible. Therefore, we have

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0. \tag{2.3.4}$$

Equation (2.3.4) is called the characteristic equation or the secular equation for the matrix $\mathbf{A}$ and the solutions $\lambda$ are the eigenvalues of $\mathbf{A}$ (or $f$).

(iii) We need to show that $\mathbf{x}_1 \cdot \mathbf{x}_2 = 0$ assuming that

$$f(\mathbf{x}_1) = \lambda_1 \mathbf{x}_1 \quad \text{and} \quad f(\mathbf{x}_2) = \lambda_2 \mathbf{x}_2 \qquad \text{with } \lambda_1 \neq \lambda_2. \tag{2.3.5}$$

$$f(\mathbf{x}_1) \cdot \mathbf{x}_2 = \lambda_1 \mathbf{x}_1 \cdot \mathbf{x}_2 \tag{2.3.6a}$$

$$\mathbf{x}_1 \cdot f(\mathbf{x}_2) = \lambda_2 \mathbf{x}_1 \cdot \mathbf{x}_2. \tag{2.3.6b}$$

Since $f$ is symmetric

$$\lambda_1 \mathbf{x}_1 \cdot \mathbf{x}_2 = \lambda_2 \mathbf{x}_1 \cdot \mathbf{x}_2. \tag{2.3.7}$$

However, $\lambda_1 \neq \lambda_2$ from which we conclude

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = 0. \tag{2.3.8}$$

(iv) (a) Consider the real and symmetric matrix

$$\mathbf{T} = \begin{pmatrix} \exp(\beta J + \beta H) & \exp(-\beta J) \\ \exp(-\beta J) & \exp(\beta J - \beta H) \end{pmatrix}. \tag{2.3.9}$$

The eigenvalues $\lambda_\pm$ of $\mathbf{T}$ are the solutions to the characteristic equation

$$\det(\mathbf{T} - \lambda \mathbf{I}) = 0. \tag{2.3.10}$$

The determinant

$$\begin{aligned}\det(\mathbf{T} - \lambda \mathbf{I}) &= \begin{vmatrix} \exp(\beta J + \beta H) - \lambda & \exp(-\beta J) \\ \exp(-\beta J) & \exp(\beta J - \beta H) - \lambda \end{vmatrix} \\ &= \lambda^2 - [\exp(\beta J + \beta H)\lambda + \exp(\beta J - \beta H)] + \exp(2\beta J) - \exp(-2\beta J) \\ &= \lambda^2 - 2\exp(\beta J)\cosh(\beta H)\lambda + \exp(2\beta J) - \exp(-2\beta J),\end{aligned} \tag{2.3.11}$$

so the solutions to the characteristic Equation (2.3.10) are

$$\begin{aligned}\lambda_\pm &= \frac{2\exp(\beta J)\cosh(\beta H) \pm \sqrt{4\exp(2\beta J)\cosh^2(\beta H) - 4[\exp(2\beta J) - \exp(-2\beta J)]}}{2} \\ &= \exp(\beta J)\left(\cosh(\beta H) \pm \sqrt{(\cosh^2(\beta H) - 1) + \exp(-4\beta J)}\right) \\ &= \exp(\beta J)\left(\cosh(\beta H) \pm \sqrt{\sinh^2(\beta H) + \exp(-4\beta J)}\right).\end{aligned} \tag{2.3.12}$$

(b) Since $\lambda_+ > \lambda_-$, the associated eigenvectors must be orthogonal. To determine the eigenvectors for $\mathbf{T}$ we must solve the equations

$$\mathbf{T}\mathbf{x}_+ = \lambda_+ \mathbf{x}_+ \tag{2.3.13a}$$

$$\mathbf{T}\mathbf{x}_- = \lambda_- \mathbf{x}_- \tag{2.3.13b}$$

or equivalently

$$(\mathbf{T} - \lambda_+ \mathbf{I})\mathbf{x}_+ = \mathbf{0} \tag{2.3.13c}$$

$$(\mathbf{T} - \lambda_- \mathbf{I})\mathbf{x}_- = \mathbf{0} \tag{2.3.13d}$$

(c) Determine explicitly the matrix $\mathsf{U}$ such that

$$\mathsf{U}^{-1}\mathsf{T}\mathsf{U} = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}. \tag{2.3.14}$$

**2.4** *Critical exponents inequality.*

Given the thermodynamic relation

$$\chi\,(C_H - C_M) = T\left(\frac{\partial\langle M\rangle}{\partial T}\right)^2_H \tag{2.4.1}$$

As $C_M \geq 0$ and $\chi \geq 0$ it follows that

$$\chi\, C_H \geq T\left(\frac{\partial\langle M\rangle}{\partial T}\right)^2_H. \tag{2.4.2}$$

Using the scaling of the different quantities close to the critical point

$$\begin{aligned}
\chi &\propto |T - T_c|^{-\gamma} && \text{for } T \to T_c,\\
C_H &\propto |T - T_c|^{-\alpha} && \text{for } T \to T_c,\\
\langle M\rangle &\propto (T_c - T)^{\beta} && \text{for } T \to T_c^- \text{ implying,}\\
\frac{\partial\langle M\rangle}{\partial T} &\propto -(T_c - T)^{\beta-1} && \text{for } T \to T_c^-
\end{aligned}$$

so by substituting into Equation (2.4.2) we find

$$\begin{aligned}
(T_c - T)^{-\gamma}\ (T_c - T)^{-\alpha} &\geq T_c\ \left(-(T_c - T)^{\beta-1}\right)^2 && \text{for } T \to T_c^-\\
(T_c - T)^{-\gamma-\alpha} &\geq T_c\ (T_c - T)^{2\beta-2} && \text{for } T \to T_c^-
\end{aligned}$$

from which we can conclude that

$$\begin{aligned}
-\gamma - \alpha \leq 2\beta - 2 &\Leftrightarrow\\
\gamma + \alpha \geq 2 - 2\beta &\Leftrightarrow\\
\alpha + 2\beta + \gamma \geq 2.&
\end{aligned} \tag{2.4.3}$$

Notice that the inequality can be repalced by an *equality* for $d = 1, 2, 3,$ and 4 and the mean-field exponents for the Ising Model.

**2.5** *The spin-spin correlation function and scaling relations.*

(i) The spin-spin correlation function

$$\begin{aligned} g(\mathbf{r}_i, \mathbf{r}_j) &= \langle\, (s_i - \langle s_i \rangle)\, (s_j - \langle s_j \rangle)\, \rangle \\ &= \langle\, s_i s_j - \langle s_i \rangle s_j - s_i \langle s_j \rangle + \langle s_i \rangle \langle s_j \rangle\, \rangle \\ &= \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle - \langle s_i \rangle \langle s_j \rangle + \langle s_i \rangle \langle s_j \rangle \\ &= \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle, \end{aligned} \tag{2.5.1}$$

where we use that the ensemble average operation $\langle \cdot \rangle$ is a linear operation and that the ensemble average of a constant is the constant itself.

(ii) Assuming that the system is translationally invariant, we substitute $m = \langle s_i \rangle = \langle s_j \rangle$ and find

$$\begin{aligned} g(\mathbf{r}_i, \mathbf{r}_j) &= \langle s_i s_j \rangle - m^2 \\ &= \langle s_j s_i \rangle - m^2 \\ &= g(\mathbf{r}_j, \mathbf{r}_i) \end{aligned} \tag{2.5.2}$$

from which it follows that the correlation function is symmetric and thus a function of the relative distance between the spins at positions $\mathbf{r}_i$ and $\mathbf{r}_j$ only, that is,

$$g(\mathbf{r}_i, \mathbf{r}_j) = g(|\mathbf{r}_i - \mathbf{r}_j|). \tag{2.5.3}$$

(iii) (a) When $|\mathbf{r}_i - \mathbf{r}_j| \to \infty$, the spins become uncorrelated, assuming that we are not at the critical point that is! Thus

$$\begin{aligned} g(\mathbf{r}_i, \mathbf{r}_j) &= \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle \\ &\to \langle s_i \rangle \langle s_j \rangle - \langle s_i \rangle \langle s_j \rangle \quad \text{for } |\mathbf{r}_i - \mathbf{r}_j| \to \infty \\ &= 0. \end{aligned} \tag{2.5.4}$$

(b) By definition the spin-spin correlation function of spin $i$ with itself

$$g(\mathbf{r}_i, \mathbf{r}_i) = \langle s_i s_i \rangle - \langle s_i \rangle \langle s_i \rangle = \langle s_i^2 \rangle - \langle s_i \rangle^2. \tag{2.5.5}$$

Because $s_i = \pm 1 \Leftrightarrow s_i^2 = 1$ we have $\langle s_i^2 \rangle = \langle 1 \rangle = 1$. Also $\langle s_i \rangle = m$, so

$$g(\mathbf{r}_i, \mathbf{r}_i) = 1 - m^2. \tag{2.5.6}$$

We assume the external magnetic field $H = 0$ so we can replace $m$ with $m_0(T)$. If $T \geq T_c$, the magnetisation $m_0 = 0$ so that

$$g(\mathbf{r}_i, \mathbf{r}_i) = \begin{cases} 1 & \text{for } T \geq T_c \\ 1 - m_0^2(T) & \text{for } T < T_c. \end{cases} \quad (2.5.7)$$

The zero-field magnetisation per spin $m_0(T) \to \pm 1$ for $T \to 0$, implying

$$g(\mathbf{r}_i, \mathbf{r}_i) \to 0 \quad \text{for } T \to 0. \quad (2.5.8)$$

This result emphasises that the correlation function measures the fluctuations of the spins *away* from the average magnetisation as is clear from the original definition

$$g(\mathbf{r}_i, \mathbf{r}_i) = \langle \, (s_i - \langle s_i \rangle)(s_j - \langle s_j \rangle) \, \rangle. \quad (2.5.9)$$

(c) In the limit $J/k_B T \ll 1$ (high temperatures relative to the coupling constant), the spins will be orientated randomly, that is, there are no correlations between the spins, so we expect $g(\mathbf{r}_i, \mathbf{r}_j) \to 0$.
In the limit $J/k_B T \gg 1$ (low temperatures relative to the coupling constant), the spins will be aligned, that is, there are no fluctuations away from the average spin, so we expect $g(\mathbf{r}_i, \mathbf{r}_j) \to 0$.

(iv) Because the susceptibility per spin diverges at the critical temperature

$$\chi(T, 0) \propto |T - T_c|^{-\gamma} \quad \text{for } T \to T_c \quad (2.5.10)$$

the volume integral of the correlation function must also diverge at the critical temperature,

$$\int_V g(\mathbf{r}) d^d\mathbf{r} \propto \int_a^\infty g(r) r^{d-1} \, dr \to \infty \quad \text{for } T \to T_c, \quad (2.5.11)$$

where $a$ is a lower cutoff = lattice constant. This implies that $g(r)$ cannot decay exponentially with distance $r$ at the critical point $T = T_c$ since this would make the integral convergent in the upper limit. However, the divergence is consistent with an algebraic decay. Assuming

$$g(r) \propto r^{-(d-2+\eta)} \quad \text{for } T = T_c, \text{ all } r = |\mathbf{r}| \tag{2.5.12}$$

then

$$\begin{aligned}
\int_V g(\mathbf{r}) d^d\mathbf{r} &\propto \int_a^\infty g(r) r^{d-1}\, dr \\
&\propto \int_a^\infty r^{-(d-2+\eta)} r^{d-1}\, dr \\
&= \int_a^\infty r^{1-\eta}\, dr \\
&= \begin{cases} \left[\frac{1}{2-\eta} r^{2-\eta}\right]_a^\infty & \text{if } \eta \neq 2 \\ [\ln(r)]_a^\infty & \text{if } \eta = 2 \end{cases}
\end{aligned}$$

that is, the integral will only diverge if the critical exponent $\eta \leq 2$. The divergence is logarithmic if $\eta = 2$ and algebraic if $\eta < 2$.

(v) (a) The correlation length diverges as $\xi(T, 0) \propto |T_c - T|^{-\nu}$ for $T \to T_c$. The critical exponent $\nu$ is independent of whether $T_c$ is approached from below or above, however, the amplitude might differ, as in the graph below. For $T > T_c$, the correlation length sets the upper linear distance over which spins are correlated. It is also identified as the linear size of the typical (characteristic) largest cluster of correlated spins and measures the typical largest fluctuation away from states with randomly oriented spins. For $T < T_c$, the correlation length measures the fluctuations away from the fully ordered state, that is, the upper linear size of the holes in the cluster of aligned spins. There will be holes on all scales up to the correlation length.

(b) When $T \neq T_c$ a finite correlation length $\xi$ is introduced and

$$g(|\mathbf{r}|) \propto r^{-(d-2+\eta)} \mathcal{G}_\pm(r/\xi) \quad \text{for } T \to T_c, \tag{2.5.13}$$

where

$$\xi \propto |T_c - T|^{-\nu} \quad \text{for } T \to T_c. \tag{2.5.14}$$

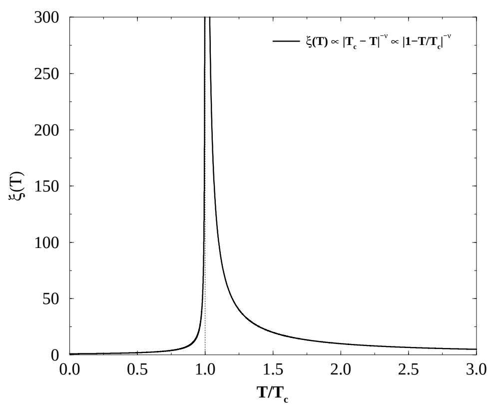



Fig. 2.5.1 The correlation length $\xi(T,0)$ as a function of the temperature $T$ in units of the critical temperature $T_c$.

Consider the relation between the susceptibility per spin and the correlation function

$$k_B T \chi \propto \int_V g(\mathbf{r}) d^d\mathbf{r}. \tag{2.5.15}$$

The left-hand side (LHS):

$$k_B T \chi \propto |T - T_c|^{-\gamma} \quad \text{for } T \to T_c. \tag{2.5.16}$$

The right-hand side (RHS):

$$\begin{aligned}\int_V g(\mathbf{r}) d^d\mathbf{r} &\propto \int_0^\infty r^{-(d-2+\eta)} \mathcal{G}_\pm(r/\xi) r^{d-1}\, dr \\ &= \int_0^\infty r^{1-\eta} \mathcal{G}_\pm(r/\xi)\, dr \\ &= \int_0^\infty (\tilde{r}\xi)^{1-\eta} \mathcal{G}_\pm(\tilde{r})\, d\tilde{r}\xi \quad \text{with } r = \tilde{r}\xi \\ &= \xi^{2-\eta} \int_0^\infty \tilde{r}^{1-\eta} \mathcal{G}_\pm(\tilde{r})\, d\tilde{r} \\ &= |T - T_c|^{-\nu(2-\eta)} \int_0^\infty \tilde{r}^{1-\eta} \mathcal{G}_\pm(\tilde{r})\, d\tilde{r} \quad \text{for } T \to T_c^\pm.\end{aligned} \tag{2.5.17}$$

The integral is just a number (which numerical value, however, depends on from which side $T_c$ is approached due to the two different scaling functions $\mathcal{G}_\pm$), so we can conclude by comparing the LHS with the RHS that

$$\gamma = \nu(2-\eta). \tag{2.5.18}$$

(c) We assume $T \le T_c$ and consider the situation in zero external field $H = 0$ with $m_0$ replacing $m$. We define

$$\tilde{g}(r) = g(r) + m_0^2 = \langle s_i s_j \rangle. \tag{2.5.19}$$

For $T < T_c$, the correlation length $\xi < \infty$. As the correlation length sets the upper limit of the linear scale over which spins are correlated, the spins will be uncorrelated in the limit $r \to \infty$ as $r \gg \xi$. Thus

$$\tilde{g}(r) = \langle s_i s_j \rangle \to \langle s_i \rangle \langle s_j \rangle = m_0^2 \propto (T_c - T)^{2\beta} \quad \text{for } T \to T_c^-. \tag{2.5.20}$$

At $T = T_c$ where the correlation length in infinite, the magnetisation is zero in zero external field, i.e., $m_0(T_c) = 0$. Thus

$$\tilde{g}(r) = g(r) \propto r^{-(d-2+\eta)} \quad \text{at } T = T_c. \tag{2.5.21}$$

One would thus expect, à la finite-size scaling in percolation theory, that

$$\tilde{g}(r) \propto \begin{cases} r^{-(d-2+\eta)} & \text{for } r \ll \xi \\ \xi^{-(d-2+\eta)} & \text{for } r \gg \xi. \end{cases} \tag{2.5.22}$$

This is the reason for considering the function $\tilde{g}(r)$ and not $g(r)$ since the latter will approach zero for $r \gg \xi$. Thus for $T < T_c$ where the correlation length is finite, we expect

$$\tilde{g}(r) \propto \xi^{-(d-2+\eta)} \propto |T - T_c|^{\nu(d-2+\eta)} \quad \text{for } r \gg \xi \tag{2.5.23}$$

implying the scaling relation

$$2\beta = \nu(d-2+\eta) \Leftrightarrow d-2+\eta = 2\beta/\nu. \tag{2.5.24}$$

**2.6** *Diluted Ising model.*

(i) A spin is situated on each lattice site. However, the spin only interacts with with the nearest neighbours with probability $p$. Identifying a nonzero coupling constant $J_{ij} = J > 0$ as an occupied bond and $J_{ij} = 0$ as an empty bond, we have an exact mapping onto a bond percolation theory problem.

(ii) (a) In order to minimise the energy, all spins within a percolation cluster will point in the same direction. However, spins belonging to different percolating clusters are not correlated.

(b) Within a cluster, $s_i = s_j$ so $s_i s_j = s_i^2 = 1$ implying $\langle s_i s_j \rangle = 1$ if the spins belong to the same cluster. If the spins $i$ and $j$ belong to different clusters, they are not correlated at all, that is, given, e.g., that $s_i = 1$ then $s_j = 1$ with probability 0.5 and $s_j = -1$ with probability 0.5 leaving $\langle s_i s_j \rangle = 0$. Hence

$$\langle s_i s_j \rangle = \begin{cases} 1 & i, j \text{ in the same percolation cluster} \\ 0 & \text{otherwise.} \end{cases} \tag{2.6.1}$$

(c) For $p < p_c$ all clusters are finite. Since the clusters are not correlated, the average magnetisation is zero. For $p > p_c$, we can argue that all the finite clusters do not contribute to the magnetisation which then becomes equal to $P_\infty(p)$, the density of the infinite cluster. The orientation of the infinite cluster is random (in zero external field). Since $P_\infty(p) = 0$ for $p < p_c$

$$m_0(p) = \pm P_\infty(p) \tag{2.6.2}$$

(iii) (a) $P_\infty(p)$ is the probability for a spin to belong to the infinite cluster. As $\tanh(sH/k_BT) \to 0$ for $H \to 0$, the last term will vanish and

$$m_0(p) = \lim_{H \to 0} m(p, H) = \pm P_\infty(p)$$

consistent with the result of (ii)(c).

(b) The susceptibility in zero external field

$$\chi = \left( \frac{\partial m}{\partial H} \right)_{H=0} .$$

Assuming $H \ll k_BT$ we use the expansion $\tanh(sH/k_BT) \approx sH/k_BT + \mathcal{O}\left((sH/k_BT)^3\right)$. Since $P_\infty(p)$ does not depend on the external field, we find,

$$\chi = \left(\frac{\partial m}{\partial H}\right)_{H=0} = \sum_{s=1}^{\infty} \frac{s^2 n(s,p)}{k_BT} \propto \chi(p) \propto |p-p_c|^{-\gamma}$$

as the divergence of the second moment of the cluster size density $n(s,p)$ is characterized by the exponent $\gamma$ when $p \to p_c$.

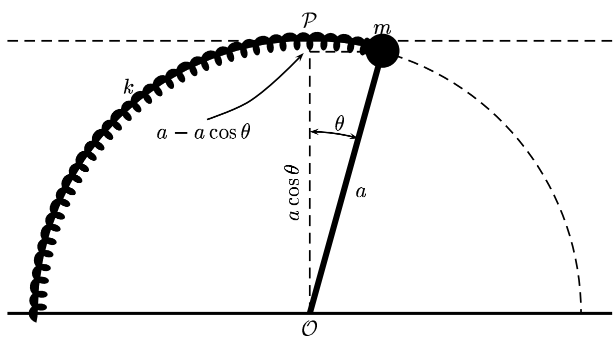
(iv) When $p < p_c$, the magnetisation in zero external field $m_0(p) = 0$. Within a cluster $\langle s_i s_j \rangle = 1$. In a cluster of size $s$ there are a total of $s^2$ different pairs, so $\frac{1}{k_BT}\sum_i \sum_j \langle s_i s_j \rangle = \frac{1}{k_BT}s^2$. We can calculate the average susceptibility by summing over all possible cluster sizes weighted by the cluster size distribution, that is,

$$\chi = \sum_{s=1}^{\infty}\left(\frac{1}{k_BT}\sum_i \sum_j \langle s_i s_j \rangle\right) n(s,p) = \frac{1}{k_BT}\sum_{s=1}^{\infty} s^2 n(s,p).$$

**2.7** *Second-order phase transition in a mass-spring system: Landau theory.*

(i) The total energy of the mass-spring system

$$\begin{aligned} U(\theta) &= \text{elastic potential energy} + \text{gravitational potential energy} \\ &= \frac{1}{2}k(a\theta)^2 + mg(a\cos\theta - a) \\ &= \frac{1}{2}ka^2\theta^2 + mga(\cos\theta - 1) \end{aligned}$$




(ii) (a) We expand the cosine to fourth order to find

$$\begin{aligned} U(\theta) &= \frac{1}{2}ka^2\theta^2 + mga(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots - 1) \\ &= \frac{a}{2}(ka - mg)\theta^2 + \frac{mga}{24}\theta^4 + \mathcal{O}(\theta^6) \end{aligned}$$

where the coefficient of the fourth-order term is positive while the coefficient of the second-order term is zero for $ka = mg$ and changes sign from positive when $ka > mg$ to negative when $ka < mg$.

(b) As the total energy $U(\theta)$ is an even function in $\theta$ (reflecting the symmetry of the problem), all the odd terms in the Taylor expansion around $\theta = 0$ are zero.

(c) When $ka > mg$, the unique minimum is at $\theta_0 = 0$. When $ka = mg$, the unique minimum is at $\theta_0 = 0$. When $ka < mg$, there are two minima at $\pm\theta_0 \neq 0$.

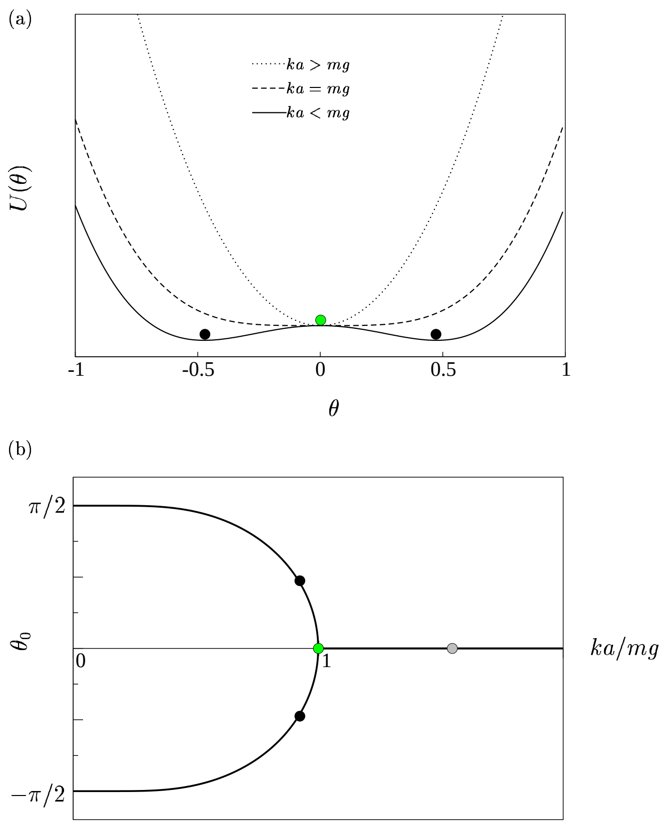



Fig. 2.7.1 (a) The energy, $U(\theta)$, versus the angle $\theta$. The solid circles show the position of the minima of the energy of the corresponding graph. For $ka > mg$, the minimal energy implies $\theta = 0$. For $ka = mg$, the trivial solution $\theta = 0$ is marginally stable However, for $ka < mg$, the minimal energy implies $\theta = \pm\theta_0 \neq 0$. (b) The angle of equilibrium, $\theta_0$ as a function of the ratio $ka/mg$.

(d) The system is in equilibrium when $dU/d\theta = 0$. Hence

$$\begin{aligned}\frac{dU}{d\theta} &= a(ka - mg)\theta + \frac{mga}{6}\theta^3 \\ &= mga\theta\left(\frac{ka}{mg} - 1 + \frac{1}{6}\theta^2\right) \\ &= 0 \qquad (2.7.1)\end{aligned}$$

with solutions

$$\theta_0 = \begin{cases} 0 & \text{for } \frac{ka}{mg} \geq 1 \\ \pm\sqrt{6(1 - ka/mg)} & \text{for } \frac{ka}{mg} < 1 \end{cases}$$
$$= \begin{cases} 0 & \text{for } \frac{m_c}{m} \geq 1 \\ \pm\sqrt{6[(m - m_c)/m]} & \text{for } \frac{m_c}{m} < 1, \end{cases}$$

where $m_c = ka/g$.

(e) See previous Figure.

(f) Landau suggested a simplistic general theory of second-order phase transitions based on expanding the free energy in powers of the order parameter. In the absence of a magnetic-like field, symmetry dictates that only even powers of the order parameter appear in the expansion. For example, in the Ising model

$$f - f_0 = a_2(T - T_c)m^2 + a_4 m^4 \quad \text{with } a_2, a_4 > 0,$$

where an expansion up to fourth order is sufficient to give a qualitative description of second-order phase transitions occurring at temperature $T_c$. The term $f_0$ is an unimportant constant, while $a_4 > 0$ in order for the free energy to be physically realistic, i.e. not minimised by extreme values of the order parameter.
As written, the left-hand side is given by a quartic polynomial which always has one trivial solution, $m = 0$, and two non-trivial solutions, $m = \pm m_0(T)$, so long as $T < T_c$. As $T$ passes through $T_c$ from above, the trivial solution becomes unstable and two stable non-trivial solutions appear. Below $T_c$, therefore, the order parameter of the system is non-zero.

(g) The order parameter of the mass-spring system is the equilibrium angle $\theta_0$ which is zero for $m \leq m_c$ and non-zero for $m > m_c$. The critical value of the variable mass $m_c = ka/g$.

**2.8** *Scaling ansatz of free energy and scaling relations.*

Consider the Ising model on a $d$-dimensional lattice in an external field $H$.

(i) (a) The total energy for a system of $N$ spins $s_i = \pm 1$ with constant nearest-neighbour interactions $J > 0$ placed in a uniform external field $H$ is

$$E_{\{s_i\}} = -J \sum_{\langle ij \rangle} s_i s_j - H \sum_{i=1}^{N} s_i,$$

where the notation $\langle ij \rangle$ restricts the sum to run over all distinct nearest-neighbour pairs.

(b) Let $M_{\{s_i\}} = \sum_{i=1}^{N} s_i$ denote the total magnetisation and $\langle M \rangle$ the average total magnetisation. The order parameter for the Ising model is defined as the magnetisation per spin

$$m(T, H) = \lim_{N \to \infty} \frac{\langle M \rangle}{N}.$$

Consider the free energy $F = \langle E \rangle - TS$. The ratio of the average total energy, $\langle E \rangle$, to the temperature times entropy, $TS$, defines a dimensionless scale $J/k_B T$. A competition exists between the tendency to randomise the orientation of spins for $J \ll k_B T$, and a tendency to align spins for $J \gg k_B T$. In the former case, the free energy is minimised by maximising the entropic term: the magnetisation is zero because the spins point up and down randomly. In the latter case, the free energy is minimised by minimising the total energy: the magnetisation is non-zero because the spins tend to align. Since the entropy in the free energy is multiplied by temperature, for sufficiently low temperatures, the minimisation of the free energy is dominated by the minimisation of the total energy. Therefore, at least qualitatively, there is a possibility of a phase transition from a phase with zero magnetisation at relatively high temperatures, to a phase with non-zero magnetisation at relatively low temperatures.

We assume that the singular part of free energy per spin is a generalised homogeneous function,

$$f(t, h) = b^{-d} f(b^{y_t} t, b^{y_h} h) \quad \text{for } t \to 0^{\pm}, h \to 0, b > 0. \qquad (2.8.1)$$

(ii) (a) The critical exponent $\alpha$ associated with the specific heat in zero external field is defined by

$$c(t,0) \propto |t|^{-\alpha} \quad \text{for } t \to 0.$$

The specific heat is related to the free energy per spin:

$$c(t,h) \propto \left(\frac{\partial^2 f}{\partial t^2}\right) \propto b^{2y_t - d} f''(b^{y_t}t, b^{y_h}h)$$

Choosing $b = |t|^{-1/y_t}$ and setting $h = 0$ we find

$$c(t,0) \propto |t|^{-\frac{2y_t - d}{y_t}} f''(\pm 1, 0) \quad \text{for } t \to 0^{\pm},$$

and we identify

$$\alpha = \frac{2y_t - d}{y_t}$$

(b) The critical exponent $\beta$ associated with the order parameter (magnetisation per spin) in zero external field is defined by

$$m(t,0) \propto |t|^{\beta} \quad \text{for } t \to 0^{-}.$$

The magnetisation per spin is related to the free energy per spin:

$$m(t,h) \propto -\left(\frac{\partial f}{\partial h}\right) \propto b^{y_h - d} f'(b^{y_t}t, b^{y_h}h).$$

Choosing $b = |t|^{-1/y_t}$ and setting $h = 0$ we find

$$m(t,0) \propto |t|^{\frac{d - y_h}{y_t}} f'(\pm 1, 0) \quad \text{for } t \to 0^{\pm},$$

and we identify

$$\beta = \frac{d - y_h}{y_t}$$

(c) The critical exponent $\gamma$ associated with the susceptibility in zero external field is defined by

$$\chi(t,0) \propto |t|^{-\gamma} \quad \text{for } t \to 0.$$

The susceptibility is related to the free energy per spin:

$$\chi(t,h) \propto -\left(\frac{\partial^2 f}{\partial h^2}\right) \propto b^{2y_h - d} f''(b^{y_t} t, b^{y_h} h).$$

Choosing $b = |t|^{-1/y_t}$ and setting $h = 0$ we find

$$\chi(t,0) \propto |t|^{-\frac{2y_h - d}{y_t}} \quad \text{for } t \to 0$$

and we identify

$$\gamma = \frac{2y_h - d}{y_t}.$$

(d) The critical exponent $\delta$ associated with the order parameter in the critical temperature is defined by

$$m(0,h) \propto \text{sign}(h)|h|^{1/\delta} \quad \text{for } h \to 0.$$

The magnetisation per spin is related to the free energy per spin:

$$m(t,h) \propto -\left(\frac{\partial f}{\partial h}\right) \propto b^{y_h - d} f'(b^{y_t} t, b^{y_h} h).$$

Choosing $b = |h|^{-1/y_h}$ and setting $t = 0$ we find

$$m(0,h) \propto |h|^{\frac{d - y_h}{y_h}} \quad \text{for } h \to 0$$

and we identify

$$\delta = \frac{y_h}{d - y_h}$$

(e) We find

$$\begin{aligned}\alpha + 2\beta + \gamma &= \frac{2y_t - d + 2d - 2y_h + 2y_h - d}{y_t} \\ &= 2\end{aligned}$$

and

$$\begin{aligned}\beta(\delta - 1) &= \frac{d - y_h}{y_t}\left(\frac{y_h}{d - y_h} - 1\right) \\ &= \frac{d - y_h}{y_t}\left(\frac{2y_h - d}{d - y_h}\right) \\ &= \frac{2y_h - d}{y_t} \\ &= \gamma.\end{aligned}$$

***Exercises***

**3.1** *Power-law probability density with exponent −2.*

(i) As $P(h) = 0$ for $h < h_{min}$, the condition for normalisation is

$$\begin{aligned} \int_{h_{min}}^{\infty} P(h)\, dh &= \int_{h_{min}}^{\infty} A h^{-2}\, dh \\ &= A[-h^{-1}]_{h_{min}}^{\infty} \\ &= A h_{min}^{-1} \\ &= 1 \end{aligned} \tag{3.1.1}$$

implying that

$$A = h_{min}. \tag{3.1.2}$$

(ii) (a) By definition, we have to integrate the probability density over $h \geq h_{max}$:

$$\begin{aligned} P(h \geq h_{max}) &= \int_{h_{max}}^{\infty} h_{min} h^{-2}\, dh \\ &= h_{min}[-h^{-1}]_{h_{max}}^{\infty} \\ &= \frac{h_{min}}{h_{max}}. \end{aligned} \tag{3.1.3}$$

(b) The average number of days one would have to wait to see one event with $h \geq h_{max}$ is

$$\frac{1}{P(h \geq h_{max})} = \frac{h_{max}}{h_{min}}.$$

(iii) (a) There is no upper limit to the level of the river, so it is impossible to guarantee safety forever.

(b) There are $365N$ days in $N$ years. The probability of having no overflow in $365N$ consecutive days is

$$\begin{aligned} P(\text{No overflow in } 365N \text{ days}) = \left(1 - \frac{h_{min}}{h_{max}}\right)^{365N} &\geq p \Rightarrow \\ 1 - \frac{h_{min}}{h_{max}} &\geq p^{\frac{1}{365N}} \Rightarrow \\ h_{max} &\geq \frac{h_{min}}{1 - p^{\frac{1}{365N}}}. \end{aligned}$$

(c) Inserting $N = 10, p = 0.90$ and $h_{min} = 0.01$ m we find

$$h_{max} \geq \frac{0.01 \text{ m}}{1 - 0.90^{\frac{1}{3650}}} \approx 346 \text{ m}.$$

(iv) (a) The average level

$$\begin{aligned}\langle h \rangle &= \int_{h_{min}}^{\infty} h_{min} h \, h^{-2} \, dh \\ &= h_{min} \int_{h_{min}}^{\infty} h^{-1} \, dh \\ &= h_{min} [\ln(h)]_{h_{min}}^{\infty} \\ &= \infty.\end{aligned} \tag{3.1.4}$$

Note that this is a so-called marginal case where the average level diverges logarithmically. A power-law probability with an exponent less than $-2$ would have a finite average value, while a power-law probability with an exponent greater than $-2$ would diverge algebraically.

(b) One could imagine that there exists an upper cut-off, $h_c$, in the level of the river for the probability density such that $P(h) = 0$ for $h \geq h_c$. Another possibility would be to modify the power-law exponent such that it is slightly less than $-2$.

**3.2** *Olami-Feder-Christensen model.*

(i) Generally we have

$$\sum_{s=1}^{\infty} s^{-a} = \begin{cases} \text{convergent} & a > 1 \\ \text{divergent} & a \leq 1 \end{cases} \tag{3.2.1}$$

Since the avalanche-size probability is normalised:

$$\sum_{s=1}^{\infty} P(s) < \infty \Rightarrow \tau_s > 1 \tag{3.2.2}$$

and the average avalanche size diverges:

$$\langle s \rangle = \sum_{s=1}^{\infty} sP(s) = \sum_{s=1}^{\infty} s^{1-\tau_s} = \infty \Rightarrow \tau_s \leq 2. \tag{3.2.3}$$

Alternatively, use the following argument

$$\sum_{s=1}^{\infty} P(s) \approx \int_1^{\infty} P(s) ds \propto [s^{1-\tau_s}]_1^{\infty} \tag{3.2.4}$$

which is only convergent in the upper limit for $\tau_s > 1$ and

$$\langle s \rangle = \sum_{s=1}^{\infty} sP(s) \approx \int_1^{\infty} sP(s) ds \propto [s^{2-\tau_s}]_1^{\infty} \tag{3.2.5}$$

which is only divergent in the upper limit for $\tau_s \leq 2$ (logarithmically so for $\tau_s = 2$).

(ii) As the cutoff event size $s_\xi$ diverges for $\alpha \to \alpha_c$,the limiting function of $P(s)$ will be a pure power law, that is,

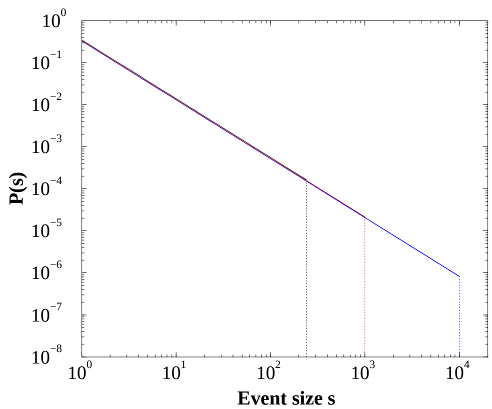
$$P(s) = \begin{cases} s^{-\tau_s} & \text{for } s \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

(iii)

$$\langle s^k \rangle = \sum_{s=1}^{\infty} s^k P(s) \approx \int_1^{\infty} s^k P(s) ds = \int_1^{s_\xi} s^{k-\tau_s} ds$$
$$\propto s_\xi^{1+k-\tau_s} \propto (\alpha_c - \alpha)^{\frac{\tau_s - k - 1}{\sigma}}$$

that is,

$$\gamma_k = \frac{\tau_s - k - 1}{\sigma}.$$




(iv) The dynamical rules of the model are motivated by the dynamics of earthquakes in which there are two separate time scales. One is defined by the motion of the tectonic plates, and the other is the duration of an earthquake. The former time scale is much larger than the latter. We separate the time scales by considering the earthquake as instantaneous, that is, the system is not driven during an earthquake.
The algorithm for the system is as following:

- Define random initial strains in the system.
- Strain is accumulated uniformly across the system as the rigid plates move.
- When the strain in a certain site is above the threshold value $F_{th}$ this site will relax according to the equation

$$\begin{aligned} F_{nn} &\to F_{nn} + \alpha F_{ij}\,, \\ F_{ij} &\to 0, \end{aligned} \tag{3.2.6}$$

  where $F_{nn}$ denote the nearest-neighbour blocks of the relaxing block $(i, j)$ and $\alpha = K/(4K + K_L)$.
  This may cause neighbouring sites to exceed the threshold value, in which case these sites relax simultaneously, and so on. The triggered earthquake will stop when there are no sites left with a strain above the threshold value.
- Strain starts to accumulate once again.

As the relaxation dissipates $F_{ij}$ but an amount of $4\alpha F_{ij}$ is redistributed, we refer to $4\alpha$ as the level of conservation.

(v) (a) The model is considered to be critical if, for a given value of $\alpha$, the event size distribution $P(s)$ is a power law with a cutoff size that diverges with systems size $L$. This will also imply that the average event size will diverge with system size. If, on the other hand, the cutoff size does not increase with system size, the model would be non-critical.
Clearly, for $\alpha = 0$, the blocks do not interact at all, and all the avalanches are of size 1, that is, the model is non-critical. For $\alpha = 0.25$, the model is conservative (conservation level = 1), and all the dissipation will take place at the boundary only. Thus one would expect the average avalanche size to diverge with system size, consistent with a power law distribution $P(s)$.
As the model is non-critical for $\alpha = 0$ and critical for $\alpha = 0.25$ there must be a crossover at some critical value $\alpha_c$ from a critical to a non-critical behaviour as $\alpha$ decreases from 0.25 to 0. Where the transition happens is still an unsettled question. There are claims that $\alpha_c = 0.25$ and $\alpha_c = 0$.

**3.3** *Modified Bak-Tang-Wiesenfeld model on a tree-like lattice.*

(i) (a) Each of the $N$ sites can be in one of $h_c$ state, $h_i = 0, 1, \ldots, h_c - 1$. Thus there are a total of $h_c^N$ stable configurations.
(b) Stable configurations are either transient or recurrent configurations. Transient configurations are not encountered once the set of recurrent configurations is reached. The set of recurrent configurations is commonly known as the attractor of the dynamics.
(c) Given a configuration in the set of recurrent states. Simply by adding $h_c - 1 - h_i$ grains to each of the $i$ sites we recover the minimally stable configuration with $h_i = h_c - 1$ for all sites $i$.
Adding one grain at the root of the tree-like structure in the minimally stable configuration will induce an avalanche in which all the grains will leave the system at the boundary and leave the system empty.
Since the empty configuration is a recurrent state, all stable configurations will be recurrent as they can be reached from the empty configuration by adding grains in a pre-

scribed way.

(ii) (a) A site with $h = h_c - 1$ will topple if it receives one grain. Such sites occur with probability $P_{h_c-1}$. Sites with $h < h_c - 1$ will not topple upon receiving one grain. Such sites occur with probability $1 - P_{h_c-1}$. Since a toppling site adds one grain to its $h_c$ downwards neighbours the probability of causing $b$ new sites to topple is determined by the binomial distribution

$$p_b = \binom{h_c}{b} P_{h_c-1}^b \left(1 - P_{h_c-1}\right)^{h_c-b} \quad b = 0, \ldots, h_c.$$

(b) The number of trials are $h_c$, each with a probability $P_{h_c-1}$ of causing a new toppling. Therefore, the average number of new topplings

$$\langle b \rangle = \sum_{b=0}^{h_c} b p_b = h_c P_{h_c-1}.$$

(iii) Since the probability $P_h$ must be normalised,

$$\sum_{h=0}^{h_c-1} P_h = h_c P_h = 1 \Leftrightarrow P_h = \frac{1}{h_c}.$$

Therefore, clearly

$$\langle b \rangle = h_c P_{h_c-1} = 1.$$

This is the critical branching ratio for a branching process. Thus the model self-organised into a critical state in which there are avalanches of all sizes, limited by the system size only.

(iv) (a) In a tree with $h_c = 2$ in a stable configuration, each site can be in one of two states, either $h_i = 0$ or $h_i = 1$. Define for now sites with $h_i = 1$ as occupied sites and sites with $h_i = 0$ as empty sites. Then the probability that a site is occupied is $P_{h=1} = 1/h_c = 1/2$, the critical occupation probability of percolation model on a Bethe lattice with $z = 3$. However, the sandpile model organises itself to this critical state.

(b) When adding a grain to an arbitrary site, it topples with probability $P_{h_c-1}$. Define $B$ to be the contribution to the average avalanche size from a given sub-branch. Then the average avalanche size is

$$\langle s \rangle = P_{h_c-1}\left(1 + h_c B\right), \qquad (3.3.1)$$

where the first term is the contribution from the toppling site itself and the second term is the contribution from the $h_c$ sub-branches. If the parent of a sub-branch has $h_i < h_c - 1$ there is no contribution. If, however, the parent of a sub-branch has $h_i = h_c - 1$, that parent contributes its own toppling together with a contribution $B$ from each of its $h_c$ subbranches. The contribution from a subbranch is identical to the contribution from a branch because all sites are equivalent. Thus

$$B = 0 \times (1 - P_{h_c-1}) + [1 + h_c B] \times P_{h_c-1}$$

from which

$$B = \frac{P_{h_c-1}}{1 - h_c P_{h_c-1}}.$$

Substituting this result into Equation (3.3.1) we find

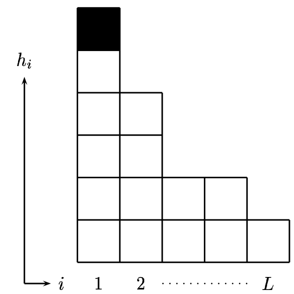
$$\langle s \rangle = P_{h_c-1}\left(1 + h_c \frac{P_{h_c-1}}{1 - h_c P_{h_c-1}}\right) = \frac{P_{h_c-1}}{1 - h_c P_{h_c-1}}$$

which diverges for $P_{h_c-1} \to 1/h_c$.

**3.4** *Oslo model and moments.*

(i) Starting from an empty system, a pile will gradually form when adding grains. However, eventually, after a transient period, the pile will cease to grow and, on average, the number of grains added at the left boundary will leave the system at the right boundary. Once the system has reached the attractor of the dynamics, the avalanches initiated by adding grains at the left boundary is only limited by the size of the system. The system has, by itself, organised into a state in which the average avalanche scales with system size, the signature of criticality.

(ii) (a) Define the local slope $z_i = h_i - h_{i+1}, i = 1, \ldots L$ with $h_{L+1} = 0$. In the one-dimensional Oslo model, the critical slopes, $z_i^c(t)$, dependent on position and time.




The algorithm for the dynamics is defined as follows.

**1.** Place the pile in an arbitrary stable configuration with $z_i \le z_i^c$ for all $i$.
**2.** Add a grain at site $i = 1$, that is, $z_1 \to z_1 + 1$.
**3.** If $z_i > z_i^c$, the site relaxes and

$$\begin{aligned} z_i &\to z_i - 2 \\ z_{i\pm1} &\to z_{i\pm1} + 1 \end{aligned}$$

except when boundary sites topple, where, respectively,

$$\begin{aligned} z_1 &\to z_1 - 2 & \qquad z_L &\to z_L - 1 \\ z_2 &\to z_2 + 1 \quad \text{for } i = 1 & \qquad z_{L-1} &\to z_{L-1} + 1 \quad \text{for } i = L. \end{aligned}$$

Choose a new critical slope $z_i^c$ at toppling site. A stable configuration is reached when $z_i \le z_i^c$ for all $i$.
**4.** Proceed to step **2.** and reiterate.

(b) The pile will eventually reach a statistically stationary state where, on average, the number of grains added will leave the system at the open boundary. Configurations are either transient configuration or recurrent configurations. Recurrent configurations will appear again and again if we wait long enough.

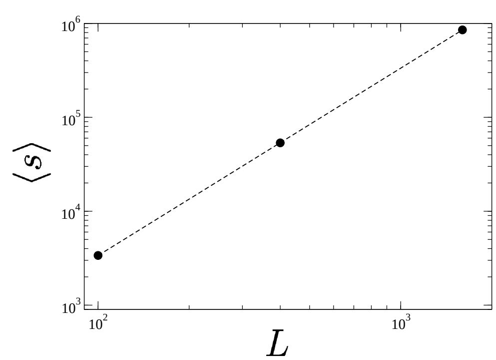
(iii) The $k$th moment

$$\begin{aligned}\langle s^k\rangle &= \sum_{s=1}^{\infty} s^k P(s,L)\\ &= \sum_{s=1}^{\infty} s^{k-\tau_s}\mathcal{G}(s/L^D)\\ &\approx \int_1^{\infty} s^{k-\tau_s}\mathcal{G}(s/L^D)ds\\ &= \int_{1/L^D}^{\infty}(uL^D)^{k-\tau_s}\mathcal{G}(u)L^D du \quad \text{with } u = s/L^D\\ &= L^{D(k+1-\tau_s)}\int_{1/L^D}^{\infty}u^{k-\tau_s}\mathcal{G}(u)du\end{aligned}$$

For $L \gg 1$, the lower limit of the integral approaches zero, and the integral becomes just a numerical factor. Therefore,

$$\begin{aligned}\langle s^k\rangle &\approx L^{D(k+1-\tau_s)}\int_0^{\infty}u^{k-\tau_s}\mathcal{G}(u)du \quad \text{for } L\gg 1\\ &\propto L^{D(k+1-\tau_s)}.\end{aligned}$$

(iv) (a) Plotting $\log\langle s\rangle$ versus $\log L$, we see that the data fall on a line with slope approximately 2.



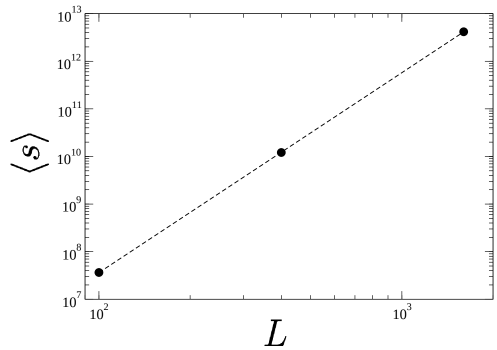

Therefore

$$\langle s\rangle \propto L^2 \propto L^{D(2-\tau_s)} \quad \text{for } L\gg 1, \tag{3.4.1}$$

implying the scaling relation

$$D(2-\tau_s) = 2. \tag{3.4.2}$$

(b) Plotting for example $\log\langle s^2\rangle$ versus $\log L$, the data fall on a line with slope approximately 4.2.




Thus

$$D(2-\tau_s) = 2 \tag{3.4.3a}$$
$$D(3-\tau_s) = 4.2 \tag{3.4.3b}$$

from which, by subtraction

$$D \approx 2.2 \tag{3.4.4}$$

and using the scaling relation in Equation (3.4.2)

$$\tau_s = 2 - 2/D \approx 1.1. \tag{3.4.5}$$

**3.5** *Moment ratios and universality.*

(i) Given that the avalanche-size probability

$$P(s;L) = as^{-\tau_s}\mathcal{G}\left(s/bL^D\right) \quad \text{for } s \gg 1, L \gg 1$$

then be rearranging we find

$$\frac{1}{a}s^{\tau_s}P(s;L) = \mathcal{G}\left(s/bL^D\right) \quad \text{for } s \gg 1, L \gg 1.$$

For a given system $a$ and $b$ are constant. The L.H.S. is a function of two variables $s$ and $L$ while the R.H.S. is a function of one variable only, $s/bL^D$. Hence by plotting the transformed avalanche-size probabilities $\frac{1}{a}s^{\tau_s}P(s;L)$ versus the rescaled avalanche size, $s/bL^D$, the data should, for $s \gg 1$ collapse onto the graph for the scaling function $\mathcal{G}$.

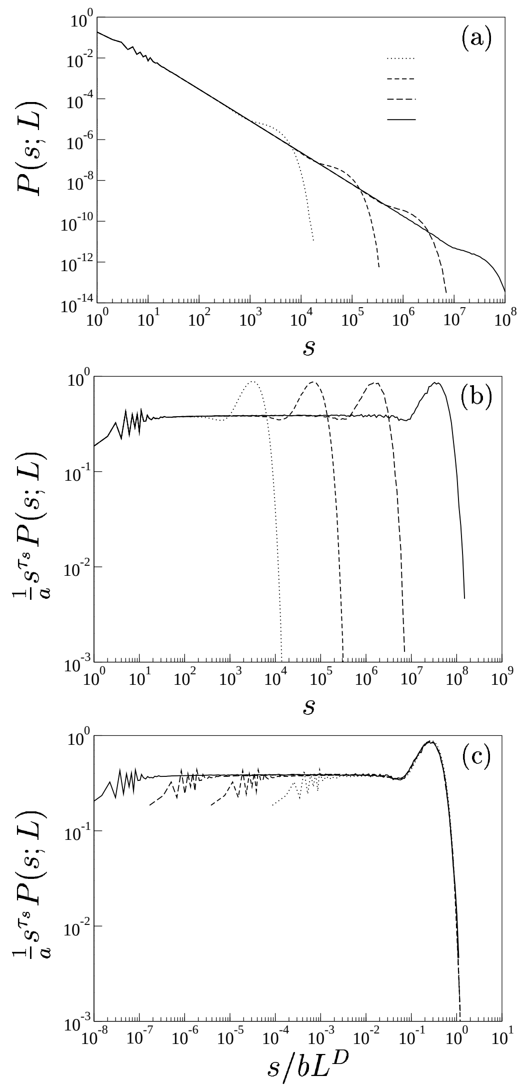



Fig. 3.5.1 (a) The avalanche-size probabilities, $P(s;L)$, versus avalanche size, $s$. The four curves correspond to lattices of increasing size marked with lines of increasing dash length. (b) The transformed avalanche-size probabilities, $\frac{1}{a}s^{\tau_s}P(s;L)$, versus avalanche size, $s$. (b) Plotting the transformed avalanche-size probability, $\frac{1}{a}s^{\tau_s}P(s;L)$, versus the rescaled avalanche size, $s/bL^D$, produces a data collapse onto a universal scaling function $\mathcal{G}$ when using the appropriate exponents $D$ and $\tau_s$.

(ii) (a) Assuming the scaling form of the avalanche-size probability is valid for all $s$ and converting the sum into an integral we find

$$\begin{aligned}
\langle s^k \rangle &= \sum_{s=1}^{\infty} s^k P(s;L) \\
&= \sum_{s=1}^{\infty} a s^{k-\tau_s} \mathcal{G}\left(s/bL^D\right) \\
&\approx \int_1^{\infty} a s^{k-\tau_s} \mathcal{G}\left(s/bL^D\right)\, ds \\
&= \int_{1/bL^D}^{\infty} a(ubL^D)^{k-\tau_s} \mathcal{G}(u)\; bL^D du \quad \text{with } u = s/bL^D \\
&= a(bL^D)^{1+k-\tau_s} \int_{1/bL^D}^{\infty} u^{k-\tau_s} \mathcal{G}(u)\; du \\
&= L^{D(1+k-\tau_s)} a b^{1+k-\tau_s} \int_0^{\infty} u^{k-\tau_s} \mathcal{G}(u)\; du,
\end{aligned}$$

since the lower limit of the integral tends to zero as $L \to \infty$. Hence we identify the universal exponent and the non-universal amplitude

$$\begin{aligned}
\gamma_k &= D(1+k-\tau_s) && \text{universal} \\
\Gamma_k &= a b^{1+k-\tau_s} \int_0^{\infty} u^{k-\tau_s} \mathcal{G}(u)\; du && \text{non-universal.}
\end{aligned}$$

(b) The moment ratio

$$g_k = \frac{\langle s^k \rangle \langle s \rangle^{k-2}}{\langle s^2 \rangle^{k-1}} = \frac{\Gamma_k L^{D(1+k-\tau_s)} (\Gamma_1 L^{D(2-\tau_s)})^{k-2}}{(\Gamma_2 L^{D(3-\tau_s)})^{k-1}} = \frac{\Gamma_k \Gamma_1^{k-2}}{\Gamma_2^{k-1}}$$

which is clearly independent of the non-universal constants $a$ and $b$.

(iii) (a) In the derivation above, we assumed the scaling form of the avalanche-size probability. However, that is only valid for $L \gg 1$. Hence only for $L \to \infty$ will the moment ratio $g_k$ be independent on system size.

(b) Model $A$ and $B$ might be in the same universality class. However, Model $C$ must belong to another universality class. Otherwise the asymptotic value of $g_3$ cannot be different.